The Geometry of Integrable Vortices

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Submitted for the degree of
Doctor of Philosophy

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May, 2019

Abstract

This thesis is concerned with geometric interpretations of vortices. We demonstrate that all five of the integrable Abelian vortex equations can be encoded in terms of the flatness and holomorphic trivialisation of a non-Abelian connection. There is a natural lift of this story to three dimensional group manifolds where the flat connection is related to the Maurer-Cartan one-form. In particular we present a detailed study of vortices on the two-sphere and on two dimensional hyperbolic space. For these cases the lifted vortices give rise to solutions of coupled equations including a massless gauged Dirac equation on flat three dimensional space. Squaring the Dirac equation arising from vortices on the 2-sphere gives rise to a Schrödinger-like equation for a spinor wave function in the background of a magnetic field with non-trivial linking. Both the wave function and the magnetic field pick up non-trivial topology related to the vortex. In the final Section a potential realisation scheme for magnetic fields with non-trivial linking as synthetic fields in a Bose-Einstein condensate is discussed.

To my family.

Acknowledgements

First up I want to thank my supervisor, Bernd Schroers, for giving generously of his time and ideas, and helping to shape the work in this thesis. Many of the ideas in this thesis are based on work that we have done in collaboration, not least the Chapters which are adapted from our joint papers.

I want to thank my examiners Richard Szabo and Derek Harland for their detailed feedback and suggestions for improvements to the first submission version of this work.

To my parents and sister I owe a debt of gratitude for their continued support and encouragement. I thank Sarah Roberts for having near infinite patience and allowing me to ramble on about work. She was also instrumental in frequent escapes from Edinburgh enabling the exploration and enjoyment of more mountainous regions.

I am grateful to many of my PhD colleagues here in MACS who, through interesting discourse on a wide range of topics, have provided a welcoming and enjoyable student experience. This manifested in a variety of forms, particularly the informal MACS PhD seminars and several reading groups run by students in the mathematical physics group. In this regard I want to single out Lennart Schmidt, with whom I shared many interesting and fruitful discussions about our respective projects and research interests. My thanks in particular go to Dermot Mc Ateer and Alec Cooper who on top of this generously gave some of their time to proofread Sections of this thesis. Their comments were a great help in trying to make some of the Chapters as accessible as possible.

Part of the work was done in collaboration with members of the Quantum Optics and Cold Atoms group in the physics department here at Heriot-Watt. In particular I would like to thank Callum Duncan who produced a couple of the figures that I use in Chapter 6.

Finally I would be remiss if I did not thank the setters of the Times quick cryptic crossword and the people behind the Times for the Times web page for having helped to provide lunch time entertainment that often allows me to feel that I have achieved something with my day. I also thank my fellow PhD students Alex

Evetts, Iain Findlay and Stuart Campbell, with whom I do battle against the daily crossword.

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Citation details	Ross, C. & Schroers, B.J. Magnetic zero-modes, vortices and Cartan geometry, Letters in Mathematical Physics 108: 949. (2018)
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Citation details	Ross, C. & Schroers, B.J. Hyperbolic vortices and Dirac fields in 2+1 dimensions, Journal of Physics A: Mathematical and Theoretical, 51, 295202. (2018)
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Chapter 1

Introduction

The core theme of this thesis is a geometric description of the five integrable Abelian vortex equations from [1] in terms of flat non-Abelian connections. This gives a complementary point of view to the metric geometry interpretation of [2]. In cases where the two-dimensional vortex may have singularities we present manifestly smooth three dimensional expressions for a lifted vortex, here called a vortex configuration. This theme is pursued through several "geometrisation" Theorems. The first of these Theorems relates integrable vortex equations on a Riemann surface to flat non-Abelian connections encoding the geometry of the Riemann surface, while the others lift this to an interpretation of vortex equations in three dimensions in terms of bundle maps between group manifolds. We encounter three such group manifolds, each of which is a circle bundle over the Riemann surface the vortex lives on.

Another recurring theme that complements the geometric description is the presence of magnetic fields with non-trivial topology. This aspect builds on and uses the work of Rañada [3, 4] who was interested in solving the Maxwell equations using the pull back of the volume form on the 2-sphere. Magnetic fields constructed in this manner can have field lines which link or form knots. Our first encounter with a magnetic field of this type is actually hidden in the integrable vortex equations on the 2-sphere where the magnetic field of the vortex can be interpreted in terms of the pull back of the volume form.

Vortices have been a subject of interest in different contexts for a long time. They arose in situations such as fluid dynamics [5] and a historical model of atoms due to Kelvin [6], which is often credited with starting the development of knot theory. They also make an appearance in the study of superconductivity [7]. The vortex equations that we encounter here are closely related to those that arise in the context of superconductivity. In the Ginsburg-Landau model [8] a superconductor is described by an energy functional for a complex scalar field, the order parameter, coupled to a gauge potential for the electromagnetic field

$$E = \frac{1}{2}\int_{R^2}\left(B^2 + |D_i\phi|^2 + \frac{\alpha}{4}\left(m^2 - |\phi|^2\right)^2\right)d^2x, \tag{1.0.1}$$

m is the vacuum expectation value of the field, $D_i = \partial_i - ia_i$ is a covariant derivative and $B = \partial_1 a_2 - \partial_2 a_1$ is the magnetic field. Depending on the value of the coupling constant α the physics of the model will be different. For $\alpha < 1$ the model describes vortices which attract while for $\alpha > 1$ it describes vortices which repel [9]. The $\alpha = 1$, critically coupled, case describes vortices which neither attract nor repel and corresponds to one of the vortex equations that we consider here.

The thesis is laid out as follows. In Chapter 2 we discuss the Abelian-Higgs model and the five integrable cases from [1]. Before giving the first of our Theorems interpreting vortices geometrically, we state and explain the conventions that we use for the Lie groups that we encounter throughout this work. They can all be interpreted as generalisations of the unit quaternions, differing by the choice of a real parameter λ; which is -1 for the unit quaternions, 1 for the unit split quaternions or 0 for the unit dual quaternions. This leads to Lie groups with a Riemannian, Lorentzian or degenerate Killing form on the Lie algebra. As we are working with Riemann surfaces throughout this thesis, we make clear our conventions for their local geometry, in particular presenting a complexified local frame, which we choose to work in, as well as the structure and Gauss equations for the surfaces. As we primarily work locally we deal only with the three surfaces: S^2, $\mathbb{R}^2 \simeq \mathbb{C}$ and H^2, two dimensional hyperbolic space. Once the conventions are out of the way, we turn our attention to vortex equations on Riemann surfaces, discuss the five integrable cases from [1] and how they are solved in terms of holomorphic maps. This leads to a discussion of the degenerate geometry defined by a vortex, due to [2], before we are ready to present the Theorem which establishes the relationship between vortices and flat connections.

Chapter 3 is a very close adaptation of the paper [10]. It takes the vortex discussion of the previous Chapter and focuses on the case of Popov vortices on the 2-sphere [11, 12], exploring their relationship with zero-modes of magnetic Dirac operators on $\mathbb{R}^3$ and with bundle maps of the Hopf fibration. We start by specialising the conventions of the previous Chapter to the group $SU(2)$ and the 2-sphere. Some time is then spent discussing the Dirac operator on $SU(2)$ and its relationship to the Dirac operator on flat Euclidean space. This story is well know from [13, 14] and we include an explicit discussion as these results are useful in the remainder of the Chapter. From there we define vortex configurations on $SU(2)$, which are the analogue of Popov vortices in three dimensions, and relate them to flat non-Abelian $SU(2)$ connections on $SU(2)$. These connections are trivialised in terms of bundle maps, $U : SU(2) \to SU(2)$. The bundle maps are the lift of the rational maps used to solve the vortex equations in [11]. Vortex configurations are then shown to give rise to vortex zero-modes of a magnetic Dirac operator on $SU(2)$. The conformal equivalence of $SU(2)$ and flat Euclidean space enables us to construct vortex magnetic modes on flat space and explore how these are related to the zero-modes from [13]. The magnetic field of this coupled Dirac operator is expressed as the superposition of two magnetic fields of the Rañada type and provides our first direct example of a magnetic field whose field lines are linked. One of these magnetic fields, called the background field throughout this thesis, has the fibres of the Hopf fibration as its field lines. Finally the relationship between our vortex configurations and Popov vortices is established and a global interpretation of the flat connections is sketched using the language of Cartan geometry. The Popov vortex story in this Chapter is slightly more general than that in Chapter 2 as the radius of the 2-sphere is left arbitrary.

Chapter 4 is another close adaptation of a paper, this time [15], and is the Lorentzian partner to Chapter 3. The story is told in the opposite direction with the familiar case of hyperbolic vortices, [9, 16], being given centre stage. We start by fleshing out some details about the treatment of hyperbolic vortex equations, particularly their solution in terms of bounded holomorphic functions from [17]. We then discuss the group structure of the double cover of AdS_3, $\widetilde{\mathrm{AdS}}_3$, the $\lambda = 1$ case of the group from Chapter 2, which plays the role here that $SU(2)$ did in the previous Chapter. Vortex configurations on $\widetilde{\mathrm{AdS}}_3$ are defined and shown to be expressible as a flat connection, now valued in the Lie algebra of $\widetilde{\mathrm{AdS}}_3$. These connections are trivialised in terms of bundle maps $V : \widetilde{\mathrm{AdS}}_3 \to \widetilde{\mathrm{AdS}}_3$, which now cover the bounded holomorphic functions of [17]. The discussion of this chapter is in some sense more general than that of the previous Chapter as there are hyperbolic vortices on more Riemann surfaces than just the disc. However, while these can in principle be lifted to vortex configurations, it is not clear how best to do this other than by a trivial lift where the bundle map is independent of the S^1 fibre. These vortex configurations are next used to construct solutions of a massless Dirac equation on $\widetilde{\mathrm{AdS}}_3$, which are pulled back to three dimensional flat Minkowski space. In contrast to the previous Chapter these Dirac modes are not defined globally but only on the inside of a single sheeted hyperboloid. This is a consequence of the stereographic projection relating $\widetilde{\mathrm{AdS}}_3$ to flat space.

Chapter 5 is a unification and generalisation of the previous Chapters. Here we return to the five integrable vortex equations of Chapter 2 and show that they can all be lifted to three dimensions in an analogous manner to the spherical and hyperbolic cases of Chapters 3 and 4. The main actors are again vortex configurations, this time defined on three dimensional group manifolds. Theorem 5.4 explains how vortex configurations are expressed in terms of bundle maps between different circle fibrations and cover holomorphic maps between constant curvature Riemann surfaces. We then move to discussing how certain vortex configurations give rise to solutions of a massless Dirac equation. This is a unifying description of the vortex magnetic modes encountered in the earlier Chapters.

At the start of each of Chapters 3, 4 and 5 we include a summary Figure sketching all of the important equations and maps used in that Chapter as well as the spaces that they are defined on. It is hoped that these provide a useful aid in reading the Chapters.

In Chapter 6 we review Rañada's method for constructing magnetic fields whose field lines include knots and links. The key to this construction is the observation, from [18], that some knots can be encoded as the preimage of infinity of a particular rational function. Some examples of magnetic fields constructed in this way are given, including that due to the composition of inverse stereographic projection and the projection from the Hopf fibration, and a map which gives rise to the trefoil knot. We then explain how the vortex magnetic modes from Chapter 3 on flat Euclidean space give rise to a zero energy, Schrödinger-like equation. By virtue of our construction of this equation, we are able to present exact solutions where both the spinor wave function and the scalar potential pick up the non-trivial topology of the Popov vortex underlying the vortex zero-mode. This model then gives a simple example of a system where we have non-trivial topology not just in the wave function but also in the background magnetic field, which is a superposition of two fields of Rañada type. The Chapter concludes with a discussion of a proposed realisation, suggested in [19], of a set up where Rañada fields can potentially be seen. This proposed realisation uses a three level system achieved in an ultracold atomic gas condensate where a synthetic magnetic field of the Rañada form arises from a Berry argument [20]. Within this scheme we adopt the method of [21–23] to approximate the linked and knotted magnetic fields with realisable fields possessing the same topology. We finish by comparing the Schrödinger equation for the condensate and the Schrödinger-like equation in the toy model of the previous Section.

Finally in Chapter 7 we summarise the results of the thesis, present some future directions and give a comparison of the flat connection picture of vortices with the seemingly closely related instanton picture given in [24]. It is hoped that this comparison will go some way to fulfil the promise to explain this relationship given at the ends of the papers [10] and [15].

Chapter 2

Vortex equations in two dimensions

2.1 The Abelian-Higgs model

The standard Abelian-Higgs model is a two dimensional model of gauged vortices. The model consists of a complex scalar field ϕ called the Higgs field and a $U(1)$ gauge potential a. On a Riemann surface M with the conformal factor Ω_0, the Abelian-Higgs model at critical coupling has the static energy functional [9]

$$E = \frac{1}{2}\int \left(\frac{B^2}{\Omega_0^2} + \frac{1}{\Omega_0}\overline{D_i\phi}D^i\phi + \frac{1}{4}\left(1-|\phi|^2\right)^2 \right) d\mathrm{Vol}, \tag{2.1.1}$$

here $B = f_{12} = \partial_1 a_2 - \partial_2 a_1$. This can be rewritten using a Bogomol'nyi argument to see that the energy is bounded below,

$$E \geq \pi|N|, \tag{2.1.2}$$

with N the winding number of the field. When $N > 0$ the minimisers solve the first order, Bogomol'nyi, equations

$$D_{\bar{z}}\phi = \left(\partial_{\bar{z}} - ia_{\bar{z}}\right)\phi = 0, \tag{2.1.3}$$

$$B = \frac{\Omega_0}{2}\left(1-|\phi|^2\right), \tag{2.1.4}$$

called the vortex equations.

For $N < 0$ the first equation, (2.1.3), becomes $D_z\phi = 0$. Now decomposing the Higgs field as $\phi = e^{u+i\chi}$, and taking account of the singularities of h at the zeros, Z_r $r = 1, \ldots, N$ possibly repeated, of the Higgs field, the Bogomol'nyi equations can be converted into the Taubes equation

$$-\frac{4}{\Omega_0}\partial_z\partial_{\bar{z}}u = 1 - e^{2u} - \frac{4\pi}{\Omega_0}\sum_{r=1}^{N}\delta(z - Z_r). \tag{2.1.5}$$

A detailed study of this equation, for the case of the Abelian-Higgs model on the plane, is in [25]. From a mathematical point of view this model is constructed from the data of a Riemann surface M, a connection a on a principle $U(1)$ bundle over M and a smooth complex section of the associated line bundle ϕ. We refer to the pair (ϕ, a) as a vortex.

Before introducing the vortex equations that we will study here we first have a brief interlude to make clear our conventions for the group theory and two-dimensional geometry that we use to talk about vortex equations.

2.2 Group conventions

We are interested in three Lie groups; $SU(2), SE_2$ and $SU(1,1)$. Respectively these are the group of determinant one, two by two unitary matrices, the component of the Euclidean group in two dimensions connected to the identity, and the group of determinant one, two by two pseudo-unitary matrices. The chosen representation of the generators of these groups is given below in (2.2.11).

The notation $\mathbb{H}^1_\lambda$ is used for all these groups as they can be viewed as generalisations of the unit quaternions. The difference coming from a factor of $\lambda = 1, 0, -1$ in one of the structure constants. We now need to understand the Lie algebra $\mathrm{Lie}\,(\mathbb{H}^1_\lambda)$.

Following [26] we introduce the parameter λ into the inverse metric as

$$g^{ab} = (-\lambda, 1, 1). \tag{2.2.1}$$

This ensures that for $\lambda = -1$ case we get the Euclidean metric and for $\lambda = 1$ the mostly plus Minkowski metric. The gamma matrices then satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2g^{ab}, \text{where } a, b = 0, 1, 2, \tag{2.2.2}$$

and we build the three Lie algebra generators, t_a, from bilinear expressions in terms of the gamma matrices as

$$t_a = \frac{1}{4}\varepsilon_{abc}\gamma^b\gamma^c. \tag{2.2.3}$$

The Lie algebra possesses a symmetric, bilinear form which is invariant under the adjoint action of the Lie group, the Killing form. To construct the Killing form we construct an inner product on the Lie algebra using Clifford multiplication, projection to the identity, Π_1, and multiplication by -4. More explicitly the inner product is constructed as

$$(t_a, t_d) = -\frac{1}{4}\varepsilon_{abc}\varepsilon_{def}\Pi_1\left(\gamma^b\gamma^c\gamma^e\gamma^f\right). \tag{2.2.4}$$

At the level of the t_a this becomes $(t_a, t_b) = -4\mathrm{Tr}\,(t_a t_b)$. This gives the Killing form as

$$\kappa_{ab} = (t_a, t_b) = \mathrm{diag}(1, -\lambda, -\lambda). \tag{2.2.5}$$

The commutation relations for the generators are thus

$$[t_a, t_b] = \varepsilon_{abc}g^{cd}t_d, \tag{2.2.6}$$

with the product of two generators being

$$t_a t_b = \frac{1}{2}\varepsilon_{abc}g^{cd}t_d + \frac{\lambda}{4}g_{ab}\mathbb{I}. \tag{2.2.7}$$

In terms of the structure constants for the Lie algebras the commutation relations are

$$[t_a, t_b] = C_{ab}{}^{c}t_c, \qquad \text{with} \quad C_{01}{}^{2} = 1, \quad C_{02}{}^{1} = -1, \quad C_{12}{}^{0} = -\lambda, \tag{2.2.8}$$

with all the others vanishing.

The parameter λ takes the value -1 for $SU(2)$, 1 for $SU(1,1)$ and 0 for SE_2. It is convenient to introduce the complex combinations

$$t_\pm = t_1 \pm t_2 \tag{2.2.9}$$

which satisfy[1]

$$[t_0, t_\pm] = \mp i t_\pm, \quad [t_+, t_-] = 2i\lambda t_0. \tag{2.2.10}$$

The first of these can be interpreted as t_0 defining a complex structure on its complement such that $(t_+)t_-$ is (anti)-holomorphic.

The following representation of the generators is chosen[2];

$$t_0 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t_1 = -\frac{i}{2}\begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}. \tag{2.2.11}$$

The group $\mathbb{H}^1_\lambda$ is a submanifold of $\mathbb{C}^2$, with the signature depending on the sign of λ,

$$\mathbb{H}^1_\lambda = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 - \lambda |z_2|^2 = 1\}. \tag{2.2.12}$$

The complex coordinates (z_1, z_2) parametrise a matrix $h \in \mathbb{H}^1_\lambda$ through

$$h = \begin{pmatrix} z_1 & \lambda \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \tag{2.2.13}$$

As $\mathbb{H}^1_\lambda$ is a Lie group, (real) left-invariant one-forms are defined through the Maurer-Cartan one-form,

$$h^{-1}dh = \sigma^0 t_0 + \sigma^1 t_1 + \sigma^2 t_2. \tag{2.2.14}$$

The one-forms obey the structure equations

$$d\sigma^a = -\frac{1}{2} C_{bc}{}^a \sigma^b \wedge \sigma^c. \tag{2.2.15}$$

[1]The commutation relations here become the same as those used for $SU(2)$ in [10] when $\lambda = -1$. However, they differ from those used for $SU(1,1)$ in [15] as there indices were raised with $(g^{ab}) = \mathrm{diag}(1, -1, -1)$ while here we use $(g^{ab}) = \mathrm{diag}(-1, 1, 1)$.

[2]The group is equivalent to G_C used in [24]. The generators, J_a, in [24] are related to the t_a used here through $t_a = -J_a^T$ with $C = \lambda$. Our conventions are picked so that when $\lambda = -1$ we match the conventions used in [10].

It is convenient to work with the complex combinations

$$\sigma = \sigma^1 + i\sigma^2, \qquad \bar{\sigma} = \sigma^1 - i\sigma^2, \tag{2.2.16}$$

which obey

$$d\sigma = i\sigma \wedge \sigma^0, \qquad d\sigma^0 = \frac{i\lambda}{2}\sigma \wedge \bar{\sigma}. \tag{2.2.17}$$

In terms of the complex coordinates the left invariant one-forms have the explicit expressions

$$\sigma = 2i\left(z_1 dz_2 - z_2 dz_1\right), \quad \sigma^0 = i\left(\bar{z}_1 dz_1 - \lambda \bar{z}_2 dz_2 - z_1 d\bar{z}_1 + \lambda z_2 d\bar{z}_2\right). \tag{2.2.18}$$

The dual left-invariant vector fields, X_a, generate the right action $h \to ht_a$ and have the commutators

$$[X_a, X_b] = C_{ab}{}^c X_c. \tag{2.2.19}$$

In terms of the combinations

$$X_\pm = X_1 \pm iX_2 \tag{2.2.20}$$

we have

$$[X_0, X_\pm] = \mp i X_\pm, \qquad [X_+, X_-] = 2\lambda i X_0. \tag{2.2.21}$$

In terms of the complex coordinates the left invariant vector fields take the form

$$X_0 = -\frac{i}{2}\left(z_1\partial_1 + z_2\partial_2 - \bar{z}_1\bar{\partial}_1 - \bar{z}_2\bar{\partial}_2\right), \tag{2.2.22}$$

$$X_- = -i\left(\bar{z}_1\partial_2 + \lambda\bar{z}_2\partial_1\right), \tag{2.2.23}$$

$$X_+ = \overline{X_-}., \tag{2.2.24}$$

where we have used $\partial_i = \frac{\partial}{\partial z_i}$.

The only non zero pairing are

$$\sigma^0(X_0) = 1, \qquad \sigma(X_-) = \bar{\sigma}(X_+) = 2. \tag{2.2.25}$$

We will make use of the group $\mathbb{H}^1_\lambda$ when we present a geometric interpretation of the integrable vortex equations in this chapter and the three dimensional geometry of $\mathbb{H}^1_\lambda$ is used in the Chapters 3, 4 and 5.

2.3 Conventions for the local geometry

Turning now to two dimensional geometry, we are primarily interested in constant curvature Riemann surfaces: hyperbolic space, H^2, the sphere, S^2, and the flat plane, $\mathbb{R}^2$. We work in the Poincaré disc model of H^2 and define it as

$$H^2 = \{z \in \mathbb{C} |\ |z|^2 < 1\}. \tag{2.3.1}$$

Before we can discuss vortices on these Riemann surfaces we need to discuss the local geometry. We work in a complexified local frame, $e, \bar{e}$ in terms of which the structure equation is

$$de - i\Gamma \wedge e = 0, \tag{2.3.2}$$

where Γ is the spin connection one-form determined by this equation. Riemann surfaces are Kähler, with the Kähler form expressed in terms of the complexified frame as

$$\omega = \frac{i}{2} e \wedge \bar{e} \tag{2.3.3}$$

The Riemann curvature two-form is

$$\mathcal{R} = d\Gamma \tag{2.3.4}$$

which satisfies the Gauss equation

$$\mathcal{R} = K\omega, \tag{2.3.5}$$

for K the Gauss curvature.

All three of $S^2, H^2, \mathbb{R}^2$ arise as the quotient of $\mathbb{H}^1_\lambda$ by a $U(1)$ subgroup,

$$M_\lambda = \mathbb{H}^1_\lambda / U(1). \tag{2.3.6}$$

This is the $U(1)$ subgroup generated by t_0 in the representation that we gave in (2.2.11).

The three different choices of λ lead to the three different surfaces

$$S^2 = \mathbb{H}^1_{-1}/U(1) = SU(2)/U(1), \tag{2.3.7}$$

$$H^2 = \mathbb{H}^1_1/U(1) = SU(1,1)/U(1), \tag{2.3.8}$$

$$\mathbb{R}^2 = \mathbb{H}^1_0/U(1) = E_2/U(1), \tag{2.3.9}$$

These three geometries are the covering spaces for Riemann surfaces. In fact the uniformisation theorem, [27], states that all simply connected Riemann surfaces are covered by the M_λ, with the covering group Λ a discrete subgroup of $\mathbb{H}^1_\lambda$. For the case of $\lambda = -1$ there is just one surface; S^2, while for $\lambda = 0$ there are three surfaces; the cylinder, the torus and $\mathbb{R}^2$, and for $\lambda = 1$ there are infinitely many Riemann surfaces realised as quotients of H^2 in this way. If $\lambda = 1$ then Λ is called a Fuchsian group.

In terms of the local complex coordinate $z \in \mathbb{C} \simeq U \subset M_\lambda$, the surface M_λ has the metric

$$ds^2_\lambda = \frac{4dzd\bar{z}}{(1-\lambda|z|^2)^2}, \tag{2.3.10}$$

a possible complexified frame for this metric is

$$e_\lambda = \frac{2dz}{1-\lambda|z|^2}. \qquad \bar{e}_\lambda = \frac{2d\bar{z}}{1-\lambda|z|^2}. \tag{2.3.11}$$

For this metric the Kähler form is

$$\omega_\lambda = \frac{2idz \wedge d\bar{z}}{(1-\lambda|z|^2)^2}. \tag{2.3.12}$$

Solving the structure equation, (2.3.2), for this frame the spin connection is found to be

$$\Gamma_\lambda = i\lambda \frac{(\bar{z}dz - zd\bar{z})}{1-\lambda|z|^2}. \tag{2.3.13}$$

The Gauss equation, (2.3.5), becomes

$$\mathcal{R}_\lambda = d\Gamma_\lambda = -\lambda\omega_\lambda, \tag{2.3.14}$$

meaning that the parameter λ can be interpreted as the negative of the Gauss curvature,

$$\lambda = -K. \tag{2.3.15}$$

All Riemann surfaces are conformally flat with the curvature dependent conformal factor Ω_λ:

$$ds^2_\lambda = \Omega_\lambda(z,\bar{z})dzd\bar{z}, \qquad \text{with } \Omega_\lambda(z,\bar{z}) = \frac{4}{(1-\lambda|z|^2)^2}. \tag{2.3.16}$$

This is the geometric framework that we use in the following section to give a geometric interpretation of vortices on particular Riemann surfaces.

2.4 Integrable vortex equations

Returning to vortex equations, in [1] a generalisation of the standard vortex equations was introduced. The key idea is to introduce two new parameters, which we call λ_0 and λ, into the vortex equations on M_{λ_0} to arrive at[3]

$$D_{\bar{z}}\phi = 0, \qquad B = \frac{\Omega_{\lambda_0}}{2}\left(\lambda_0 - \lambda|\phi|^2\right). \tag{2.4.1}$$

When we are able to write down an explicit solution for (ϕ, a) we call the vortex equations integrable. In fact all we need is an expression for $|\phi|$ then a choice of gauge and using (2.1.3) leads to (ϕ, a).

By decomposing the Higgs field as $\phi = e^{u+i\chi}$ a generalisation of the Taubes equation can be arrived at,

$$-\frac{4}{\Omega_{\lambda_0}}\partial_z\partial_{\bar{z}}u = \left(\lambda_0 - \lambda e^{2u}\right) - \frac{4\pi}{\Omega_{\lambda_0}}\sum_{r=1}^{N}\delta(z - Z_r) \tag{2.4.2}$$

where Ω_{λ_0} is the conformal factor relating the metric on M_{λ_0} to the flat metric on

[3]In [24] and [1] they use $C = -\lambda$ and $C_0 = -\lambda_0$ as the constants.

$\mathbb{R}^2$ and the Z_r are the zeros of the Higgs field.

A scaling argument is used in [1] to restrict λ_0 and λ to the values $-1, 0, 1$ giving nine possible equations. Following [1] this is reduced further to five cases by observing that the left hand side of (2.4.2) is the magnetic field and thus upon integration becomes the flux, $2\pi N$. Now as we are working with vortices which satisfy $D_{\bar{z}}\phi = 0$ we know that N is positive and this constrains λ and λ_0 such that the right hand side is also positive.

The five remaining cases are:

1. Hyperbolic vortices, $\lambda_0 = \lambda = 1$;
2. Popov vortices, $\lambda_0 = \lambda = -1$;
3. Jackiw-Pi vortices, $\lambda_0 = 0, \lambda = -1$;
4. Ambjorn-Olsen vortices, $\lambda_0 = 1, \lambda = -1$;
5. Bradlow vortices, $\lambda_0 = 1, \lambda = 0$.

In [17] it was shown that for hyperbolic vortices on H^2 the Taubes equation can be reduced to the Liouville equation. This leads to solutions of the vortex equations in terms of a bounded holomorphic map $f : H^2 \to H^2$. In terms of this map the modulus of the Higgs field is found to be [17]

$$|\phi| = \frac{1 - |z|^2}{1 - |f|^2} \left| \frac{df}{dz} \right|. \tag{2.4.3}$$

This result can be generalised to the other vortex equations in the following way. Consider the vortex equations (2.4.1) on the Riemann surface M_{λ_0} with Gauss curvature $K_0 = -\lambda_0$. Define a new function g as

$$u = g + \log\left(\frac{1}{2}\left(1 - \lambda_0|z|^2\right)\right), \tag{2.4.4}$$

then observe that

$$-4\partial_z\partial_{\bar{z}}u = -4\partial_z\partial_{\bar{z}}g + \lambda_0\Omega_{\lambda_0}, \tag{2.4.5}$$

This reduces the generalised Taubes equation, (2.4.2), to

$$4\partial_z\partial_{\bar{z}}g = \lambda e^{2g} + 4\pi\sum_{r=1}^{N}\delta(z - Z_r). \tag{2.4.6}$$

This is the Liouville equation with sources for a metric with conformal factor e^{2g} and Gauss curvature $K = -\lambda$. On a simply connected domain this is solved in terms of a meromophic function $f : M_{\lambda_0} \to M_\lambda$ [1] with $e^{2g} = f^*\Omega_\lambda \left|\frac{df}{dz}\right|^2$ and

$$|\phi| = e^u = \frac{e^g}{\sqrt{\Omega_{\lambda_0}}} = \frac{1-\lambda_0|z|^2}{1-\lambda|f|^2}\left|\frac{df}{dz}\right|. \tag{2.4.7}$$

It is instructive here to show how to extract ϕ and a from f explicitly. Picking the gauge where the phase of ϕ is the same as that of $\frac{df}{dz}$ and using that, away from the zeros of ϕ, (2.1.3) can be inverted as $a_{\bar{z}} = -i\partial_{\bar{z}} \ln \phi$ we find that

$$\phi = \frac{1-\lambda_0|z|^2}{1-\lambda|f|^2}\frac{df}{dz}, \qquad a_{\bar{z}} = -i\partial_{\bar{z}} \ln\left(\frac{1-\lambda_0|z|^2}{1-\lambda|f|^2}\right). \tag{2.4.8}$$

A consequence of this is that all five of the vortex equations from [1] are integrable on a constant curvature surface M_{λ_0} with Gauss curvature $-\lambda_0$ and are solved in terms of a meromorphic map between M_{λ_0} and another Riemann surface M_λ with constant Gauss curvature $K = -\lambda$. Two well studied cases are those of hyperbolic vortices on Riemann surfaces with hyperbolic metrics, [16, 28] and Popov vortices on S^2, [11, 12] where the vortex equations are solved in terms of a rational map $f : S^2 \to S^2$. The other integrable cases are Jackiw-Pi vortices on flat Riemann surfaces and Bradlow and Ambjorn-Olsen vortices on hyperbolic Riemann surfaces.

To get the most out of this picture it is worth using the following coordinate invariant definition of vortices and vortex equations.

Definition 2.1

A (λ_0, λ) vortex is a pair (ϕ, a) of a connection, a, and a section, ϕ, of the associated line bundle of a, possibly trivial, principal $U(1)$ bundle over M_{λ_0} which satisfy the (λ_0, λ) vortex equations

$$(d\phi - ia\phi) \wedge e_{\lambda_0} = 0 \qquad F_a = da = \left(\lambda_0 - \lambda|\phi|^2\right)\omega_{\lambda_0}. \tag{2.4.9}$$

From this point on we will be referring to (2.4.9) when we discuss vortex equations.

2.5 Vortex equations and Baptista metrics

Another way to interpret these vortex equations as being integrable is given in [24] where the authors show that the vortex equations can be realised as the dimensional reduction of Anti-self dual Yang-Mills theory on the four manifold $M_{\lambda_0} \times M_{-\lambda_0}$ with a careful choice of gauge group. This is achieved by imposing equivariance under the symmetry group of $M_{-\lambda_0}$, $\mathbb{H}^1_{-\lambda_0}$, on $\mathbb{H}^1_{-\lambda}$-instantons and results in the (λ_0, λ) vortex equations on M_{λ_0}. This is a generalisation of the construction of hyperbolic vortices as $SO(3)$ invariant instantons in [17]. We will return to comment on this dimensional reduction picture in the Conclusion where we can point out some similarities between the instanton and the flat connection picture of a vortex that we shall develop in this thesis.

The idea of interpreting vortices geometrically stems from [16] where the Higgs field of a Hyperbolic vortex was represented as the ratio of the confomal factor of the Riemann surface M_λ pulled back to M_{λ_0}, to the conformal factor of M_{λ_0}

$$|\phi|^2 = \frac{f^*\Omega_\lambda}{\Omega_{\lambda_0}} \left|\frac{df}{dz}\right|^2 \tag{2.5.1}$$

Then in [2] it was shown that the vortex could be thought of as defining a degenerate conical geometry on M_{λ_0} where the metric has the conformal factor $|\phi|^2$, which is zero at the vortex centres. This metric is referred to as the Baptista metric in [1], where the concept was extended to all five types of vortices. For the integrable cases the Baptista metric is the pullback of the metric on M_λ to M_{λ_0} by the map f,

$$ds_B^2 = f^* ds_\lambda^2 = |\phi|^2 ds_{\lambda_0}^2. \tag{2.5.2}$$

In fact this allows us to interpret the vortex as taking a complex orthogonal frame on M_λ, $e_\lambda, \bar{e}_\lambda$, and arriving at the degenerate frame $f^*e_\lambda = \phi e_{\lambda_0}, f^*\bar{e}_\lambda = \bar{\phi}\bar{e}_{\lambda_0}$. In this new, degenerate, frame the vortex equations are part of the structure and Gauss equations.

As shown above solutions to the (λ_0, λ) vortex equations are constructed from a holomorphic map $f : M_{\lambda_0} \to M_\lambda$. The Baptista metric (2.5.2) suggests the following very direct way to construct the pair (ϕ, a) from the data of the holomorphic map f

and the geometric data of the frames and spin connections on M_λ and M_{λ_0}. Define a Higgs field and a gauge potential through

$$a = f^*\Gamma_\lambda - \Gamma_{\lambda_0}, \qquad f^*e_\lambda = \phi e_{\lambda_0}. \tag{2.5.3}$$

It follows that

$$f^*\left(e_\lambda \wedge \bar{e}_\lambda\right) = |\phi|^2 e_{\lambda_0} \wedge \bar{e}_{\lambda_0}, \tag{2.5.4}$$

so that

$$F_a = d\left(f^*\Gamma_\lambda - \Gamma_{\lambda_0}\right) = f^*\mathcal{R}_\lambda - \mathcal{R}_{\lambda_0} = \left(\lambda_0 - \lambda|\phi|^2\right)\omega_{\lambda_0}, \tag{2.5.5}$$

is an immediate consequence.

To check that the first vortex equation is satisfied consider the pullback of the structure equation

$$0 = df^*e_\lambda - if^*\Gamma_\lambda \wedge f^*e_\lambda, \tag{2.5.6}$$

$$= \left(de_{\lambda_0} - i\Gamma_{\lambda_0} \wedge e_{\lambda_0}\right)\phi + \left(d\phi - i\left(f^*\Gamma_\lambda - \Gamma_{\lambda_0}\right)\phi\right) \wedge e_{\lambda_0}, \tag{2.5.7}$$

$$= \left(d\phi - ia\phi\right) \wedge e_{\lambda_0}. \tag{2.5.8}$$

The final line is the first vortex equation. In [10] and [15] this was the approach used to explore another geometric interpretation for Popov and hyperbolic vortex equations respectively. The results in those papers were that the vortex equations could be interpreted as the flatness condition of a Cartan connection. In the following section that result is generalised to all of the (λ_0, λ) vortex equations.

It is important to be aware that since the Baptista metric has the conformal factor $|\phi|^2$ it is degenerate at the N, not necessarily distinct, zeros of the Higgs field. As observed in [2] the Riemann curvature 2-form associated with the metric is extended to the zeros by adding delta function singularities

$$\mathcal{R}' = \mathcal{R}_{\lambda_0} + F_a - 2\pi\sum_{j=1}^{N}\delta_{Z_j}, \tag{2.5.9}$$

where we use δ_{Z_j} for the two-form Dirac delta supported on the point Z_j.

This can be understood as the Baptista metric having a conical singularity with surplus angle $2\pi N_j$ at a zero of multiplicity N_j, with $N = \sum_j N_j$. The local geometry around the point Z_j thus resembles a ruffled collar and is sometimes called an Elizabethan geometry. For the case of Popov vortices on S^2, a is a connection on a line bundle of even degree, $N = 2n - 2$ with $n = 1, 2, \dots$ etc, and thus

$$\int_{S^2} F_a = 4\pi n - 4\pi. \tag{2.5.10}$$

This is cancelled by the integral over the delta functions and thus

$$\int_{S^2} \mathcal{R}' = \int_{S^2} \mathcal{R}_{-1} = 4\pi, \tag{2.5.11}$$

which can be interpreted as the Gauss-Bonnet theorem holding for $\mathcal{R}'$. This is in contrast to the pullback of the curvature two-form, $f^*\mathcal{R}_{-1}$, which integrates to $4\pi n$ since in this case the map $f : S^2 \to S^2$ has degree n [1, 10].

For the other vortices we still have Equation (2.5.9) and F_a still integrates to $2\pi N$, once the appropriate boundary conditions are taken in to account, which is again cancelled by the delta function contribution. However, as H^2 and $\mathbb{R}^2$ are non-compact the integrals of the curvature forms are not defined.

This example shows that the spin connection of the degenerate frame ϕe_{λ_0}, $\tilde{\Gamma}$ differs from the pulled back spin connection $f^*\Gamma_\lambda$ by a contribution due to the zeros of ϕ and that this contribution is what leads to the singularities in $\mathcal{R}'$.

We come back to some specific discussions of the singularities in degenerate frames due to vortices in later Chapters where we specialise to the case of Popov vortices and hyperbolic vortices.

2.6 Vortices and Cartan connections

We make use of the language of Cartan geometry, see the book [29] or the thesis [30] for a review, to express the structure and Gauss equations of our model spaces in terms of flatness of a connection. Then we show how vortex equations can be encoded as the flatness of a connection.

The idea is to build a gauge potential for a connection in terms of the local frame and the spin connection, valued in the Lie algebra of $\mathbb{H}^1_\lambda$. The flatness of this connection can then be shown to be equivalent to the structure and Gauss equation for the frame. The following Proposition expresses how this is done.

Proposition 2.2

The structure and Gauss equations, (2.3.2) *and* (2.3.5)*, for the frame,* (2.3.11)*, and spin connection* (2.3.13)*, are equivalent to the flatness of the* $Lie(\mathbb{H}^1_\lambda)$ *valued connection*

$$\hat{A} = -\Gamma_\lambda t_0 + \frac{i}{2}\left(e_\lambda t_- - \bar{e}_\lambda t_+\right). \tag{2.6.1}$$

The proof is a straightforward computation of the curvature of $\hat{A}$.

Proof. The curvature of $\hat{A}$ is

$$F_{\hat{A}} = d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}], \tag{2.6.2}$$

$$= -\left(\mathcal{R}_\lambda + \lambda\omega_\lambda\right) t_0 + \frac{i}{2}\left(de_\lambda - i\Gamma_\lambda \wedge e_\lambda\right) t_- - \frac{i}{2}\left(d\bar{e}_\lambda + i\Gamma_\lambda \wedge \bar{e}_\lambda\right) t_+. \tag{2.6.3}$$

The vanishing of the coefficient of t_0 is equivalent to the Gauss equation, (2.3.5) with curvature $K = -\lambda$, and the vanishing of the $t_\pm$ coefficients is equivalent to the structure equations, (2.3.2). □

Now that the structure and Gauss equations can be interpreted as the flatness condition for a $Lie(\mathbb{H}^1_\lambda)$ connection we can take the next step and show that the (λ_0, λ) vortex equations can also be interpreted as a flatness condition for a $Lie(\mathbb{H}^1_\lambda)$ valued connection.

In [10] and [15] respectively Popov and hyperbolic vortices were shown to be equivalent to flatness of a Cartan connection valued in $su(2)$ and $su(1,1)$ respectively. Cartan connections are naturally used to describe the geometry of homogeneous spaces. The methods used for these two cases can be extended to find flat Cartan connections which are equivalent to the other vortex equations. This method is to first find a connection on M_λ, flatness of which is equivalent to the structure and Gauss equations of the complex frame for M_λ. Next the map f is used to pull back this connection to M_{λ_0} and it is checked that flatness of this pullback connection is equivalent to the desired vortex equations. This approach works because we know from [2] that $|\phi|^2$ can be interpreted as the conformal factor of the Baptista metric (2.5.2) and the vortex equations can then be interpreted as the structure and Gauss equations in the pullback frame.

Another way to say this is that if we start with a connection which encodes the geometry of M_λ then pulling back by f results in a connection which encodes the geometry of the new degenerate frame defined by the vortex pair (ϕ, a) on M_{λ_0}.

The following Theorem contains the results of Lemma 4.2 in [10] and Proposition 2.1 in [15] as special cases.

Theorem 2.3

Given the flat connection $\hat{A}$ defined as in (2.6.1)*, for the frame and spin connection $e_\lambda, \Gamma_\lambda$ on M_λ and a holomorphic map $f : M_{\lambda_0} \to M_\lambda$ the flatness of the pulled back connection $f^*\hat{A}$ is equivalent to the (λ_0, λ) integrable vortex equations on M_{λ_0}.*

This result is the two dimensional version of results that we will return to several times in this thesis, namely Theorems 3.4, 4.3 and 5.4. We will address the exact relationship between the flat connection considered here and that of the other theorems in their respective chapters. Again the method of proof is by direct computation.

Proof. From Proposition 2.2 we know that flatness of $\hat{A}$ is equivalent to the structure and Gauss equations on M_λ. This means that as f is smooth the pulled back connection, $f^*\hat{A}$, is also flat. Using (2.5.3) we have

$$f^*\hat{A} = -\left(a + \Gamma_{\lambda_0}\right) t_0 + \frac{i}{2}\left(\phi e_{\lambda_0} t_- - \bar{\phi}\bar{e}_{\lambda_0} t_+\right), \tag{2.6.4}$$

computing the curvature gives

$$\begin{aligned} F_{f^*\hat{A}} = &-\left(da - \left(\lambda_0 - \lambda|\phi|^2\right)\omega_{\lambda_0}\right) t_0 + \frac{i}{2}\left(d\phi - ia\phi\right)\wedge e_{\lambda_0} t_- \\ &- \frac{i}{2}\left(d\bar{\phi} + ia\bar{\phi}\right)\wedge \bar{e}_{\lambda_0} t_+. \end{aligned} \tag{2.6.5}$$

The (λ_0, λ) vortex equations, (2.4.9), are equivalent to the vanishing of this curvature. □

In Chapter 5 we will return to this Cartan picture and show how it can be lifted to $\mathbb{H}^1_\lambda$.

Chapter 3

The Euclidean Story

This chapter is an adaptation of the paper [10] and explores the relationship between a family of magnetic-zero-modes on $\mathbb{R}^3$ and Popov vortices. Pieces of this story were found before in [31, 32], while the relationship between magnetic zero-modes on S^3 and S^2 was explored in [14]. This family of magnetic zero-modes are examples of the type of zero-modes constructed in [13]. Following [10] the story begins by considering the Dirac operator on $SU(2)$ and its relation to the Dirac operator on $\mathbb{R}^3$, before going on to discuss vortex configurations on $SU(2)$ and explain how these lead to magnetic zero-modes on $SU(2)$. Finally, the relationship between vortex configurations and Popov vortices, the $\lambda_0 = \lambda = -1$ vortices from Chapter 2, will be explored. Figure 3.1 encapsulates all the equations that will be discussed in this Chapter.

3.1 Magnetic Dirac operators on S^3 and on $\mathbb{R}^3$

3.1.1 Conventions for $SU(2)$ and the Hopf map

The conventions used in this chapter are the same as those used in the paper [10] and follow those of the previous chapter specialised to the case of $\lambda = \lambda_0 = -1$. In this chapter we use Latin indices i, j, k which run over the range $1, 2, 3$. The $su(2)$ generators used here are

$$t_1 = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t_3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.1.1}$$

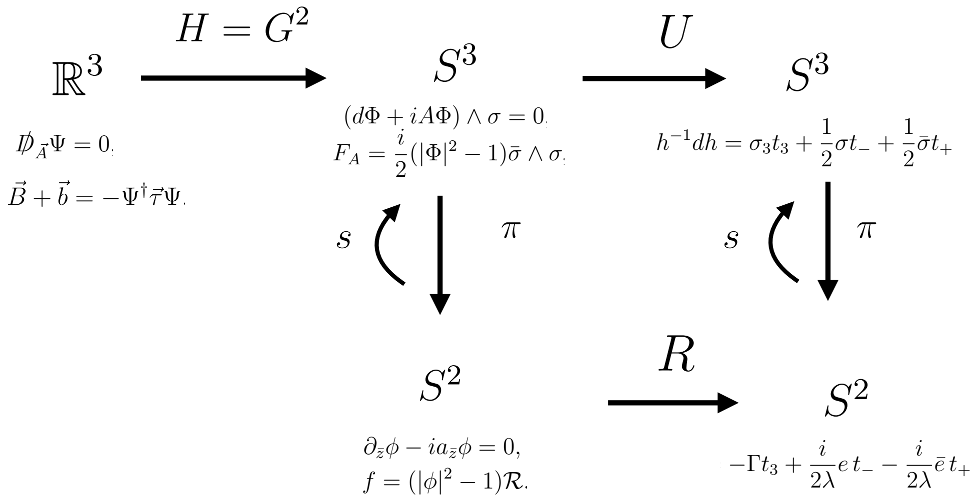

Figure 3.1: A summary of the equations and maps studied in this chapter, The Figure was produced by B. Schroers.

with commutators $[t_i, t_j] = \epsilon_{ij}{}^k t_k$. As mentioned in the previous chapter we will often work with the complex combinations $t_\pm = t_1 \pm it_2$ defined in Equation (2.2.9) which have the commutators

$$[t_3, t_+] = -it_+, \quad [t_3, t_-] = it_-, \quad [t_+, t_-] = -2it_3. \tag{3.1.2}$$

The parameterisation of a $\mathbb{H}^1_\lambda$ matrix specialises to parametrise an $SU(2)$ matrix h in terms of a pair of complex numbers (z_1, z_2) via

$$h = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \tag{3.1.3}$$

with the constraint in Equation (2.2.12) becoming the familiar condition that $\det h = |z_1|^2 + |z_2|^2 = 1$. The (real) left-invariant 1-forms are defined via

$$h^{-1}dh = \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 t_3, \tag{3.1.4}$$

and satisfy $d\sigma_1 = -\sigma_2 \wedge \sigma_3$ and the cyclic permutations of this equation. Defining

$$\sigma = \sigma_1 + i\sigma_2, \qquad \bar{\sigma} = \sigma_1 - i\sigma_2 \tag{3.1.5}$$

we also note that

$$d\sigma = i\sigma \wedge \sigma_3, \quad d\sigma_3 = \frac{i}{2}\bar{\sigma} \wedge \sigma, \tag{3.1.6}$$

and have the following expressions in terms of the complex coordinates:

$$\sigma = 2i(z_1 dz_2 - z_2 dz_1), \quad \sigma_3 = 2i(\bar{z}_1 dz_1 + \bar{z}_2 dz_2). \tag{3.1.7}$$

The dual vector fields X_j, $j = 1, 2, 3$, generate the right-action $h \mapsto ht_j$ [33]. Their commutators are

$$[X_i, X_j] = \epsilon_{ijk} X_k, \tag{3.1.8}$$

so that, in terms of

$$X_+ = X_1 + iX_2, \qquad X_- = X_1 - iX_2, \tag{3.1.9}$$

we have

$$[X_+, X_-] = -2iX_3, \qquad [iX_3, X_\pm] = \pm X_\pm. \tag{3.1.10}$$

Explicitly in terms of complex coordinates, the right-generators are

$$X_+ = i(z_1 \bar{\partial}_2 - z_2 \bar{\partial}_1), \tag{3.1.11}$$

$$X_- = i(\bar{z}_2 \partial_1 - \bar{z}_1 \partial_2), \tag{3.1.12}$$

$$X_3 = \frac{i}{2}(\bar{z}_1 \bar{\partial}_1 + \bar{z}_2 \bar{\partial}_2 - z_1 \partial_1 - z_2 \partial_2). \tag{3.1.13}$$

There is a corresponding left action[1], $h \mapsto -t_j h$, with generators the right invariant vector fields $Z_\pm$ and Z_3; expressions in terms of the complex coordinates used here are given in [33]. The Laplace operator on S^3 (with radius 2) can be written in terms of the left-invariant vector fields as

$$\Delta_{S^3} = X_1^2 + X_2^2 + X_3^2 = X_+ X_- + iX_3 + X_3^2 = X_- X_+ - iX_3 + X_3^2, \tag{3.1.14}$$

with an analogous expression in terms of the right-invariant vector fields. The only

[1]The convention for the left action may look slightly odd at first but it is picked so that the right invariant forms satisfy $[Z_i, Z_j] = \varepsilon_{ij}^{\ \ k} Z_k$.

non-zero pairings between the left-invariant one forms and vector fields are

$$\bar{\sigma}(X_+) = \sigma(X_-) = 2, \quad \sigma_3(X_3) = 1. \tag{3.1.15}$$

In this Chapter we parametrise the 2-sphere via stereographic projection in terms of a local coordinate $z \in \mathbb{C}$. We work in the coordinates provided by stereographic projection from the south pole, This is the same choice as was made in [33] where the authors give the coordinate changes required to cover the south pole itself by projecting from the north pole. In terms of the complex coordinates (3.1.3) for S^3, the Hopf map is

$$\pi : S^3 \to S^2, \quad h \mapsto z = \frac{z_2}{z_1}. \tag{3.1.16}$$

Then a local section of the Hopf bundle is

$$s : \mathbb{C} \to SU(2), \qquad z \mapsto \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix}. \tag{3.1.17}$$

This will be used to switch from the equivariant description of sections of associated line bundles to local expressions for such sections. Defining the space of equivariant functions

$$C^\infty(S^3, \mathbb{C})_N = \{F : S^3 \to \mathbb{C} | 2iX_3 F = NF\}, \quad N \in \mathbb{N}^0, \tag{3.1.18}$$

and writing H for the hyperplane bundle over S^2, the following commuting diagram is constructed [33, 34]:

$$\begin{array}{ccc} C^\infty(S^3, \mathbb{C})_N & \xrightarrow{X_+} & C^\infty(S^3, \mathbb{C})_{N+2} \\ \Big\downarrow{\scriptstyle s^*} & & \Big\downarrow{\scriptstyle s^*} \\ C^\infty(H^N) & \xrightarrow{i((1+|z|^2)\bar{\partial}_z + \frac{N}{2} z)} & C^\infty(H^{N+2}). \end{array} \tag{3.1.19}$$

In Appendix B we give a generalisation of this result to equivariant functions on $\mathbb{H}^1_\lambda$.

3.1.2 Stereographic projection and frames

In this Chapter we consider a 2-sphere of radius R and a 3-sphere of radius ℓ. Then

$$\left(\frac{\ell}{2}\sigma_1, \frac{\ell}{2}\sigma_2, \frac{\ell}{2}\sigma_3\right) \tag{3.1.20}$$

is an orthonormal frame for S^3, and the metric in this frame is

$$ds^2 = \frac{\ell^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2). \tag{3.1.21}$$

To make contact with the standard orientation on $\mathbb{R}^3$, after stereographic projection, we define the orientation on S^3 in terms of the volume element

$$\Omega = \frac{\ell^3}{8}\sigma_2 \wedge \sigma_1 \wedge \sigma_3. \tag{3.1.22}$$

We write elements of $\mathbb{R}^3$ as $\vec{x} = (x_1, x_2, x_3)^t$, denote their length by $r = |\vec{x}|$, and use that

$$(dx_1, dx_2, dx_3) \tag{3.1.23}$$

provides an oriented orthonormal frame. In this frame the metric and volume element are

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2, \qquad dx_1 \wedge dx_2 \wedge dx_3. \tag{3.1.24}$$

Thinking of the 3-sphere of radius ℓ embedded in $\mathbb{R}^4$ with coordinates (y_1, y_2, y_3, y_4), the stereographic projection from the south pole onto $\mathbb{R}^3$ is

$$\begin{aligned} &\mathrm{St} : S^3 \setminus \{\text{south pole}\} \to \mathbb{R}^3, \\ &(y_1, y_2, y_3, y_4) \mapsto (x_1, x_2, x_3) = \left(\frac{\ell y_1}{\ell + y_4}, \frac{\ell y_2}{\ell + y_4}, \frac{\ell y_3}{\ell + y_4}\right), \end{aligned} \tag{3.1.25}$$

with inverse

$$\begin{aligned} &\mathrm{St}^{-1} : \mathbb{R}^3 \to S^3, \\ &(x_1, x_2, x_3) \mapsto (y_1, y_2, y_3, y_4) = \frac{\ell}{\ell^2 + r^2}(2x_1\ell, 2x_2\ell, 2x_3\ell, \ell^2 - r^2). \end{aligned} \tag{3.1.26}$$

Writing $\vec{t} = (t_1, t_2, t_3)^T$ for the vector whose components are the Lie algebra generators, and identifying $(y_1, y_2, y_3, y_4) \in S^3$ with the unitary matrix $(y_4\mathbb{I} - 2y_1t_1 - 2y_2t_2 - 2y_3t_3)/\ell$, the inverse stereographic projection is, up to an overall scaling by ℓ, the map

$$\begin{aligned} H &: \mathbb{R}^3 \to SU(2), \\ \vec{x} &\mapsto \frac{\ell^2 - r^2}{\ell^2 + r^2}\mathbb{I} - \frac{4\ell}{\ell^2 + r^2}\vec{x} \cdot \vec{t}, \\ &= \frac{1}{\ell^2 + r^2}\begin{pmatrix} \ell^2 - r^2 + 2i\ell x_3 & 2i\ell(x_1 - ix_2) \\ 2i\ell(x_1 + ix_2) & \ell^2 - r^2 - 2i\ell x_3 \end{pmatrix}, \end{aligned} \tag{3.1.27}$$

such that the Hopf projection in stereographic coordinates is

$$\pi \circ H : (x_1, x_2, x_3) \mapsto \frac{2\ell(x_1 + ix_2)}{2x_3\ell + i(r^2 - \ell^2)}. \tag{3.1.28}$$

When pulling back objects from S^3 to $\mathbb{R}^3$ we will make use of H rather than St^{-1} to simplify the notation. This is despite the fact that the two maps are valued in 3-spheres of different radii.

A recurring theme in this Chapter is the interplay between the stereographic projection and the gnomonic projection, often used in cartography, which maps great circles to straight lines. The inverse of the gnomonic projection is the map

$$G : \mathbb{R}^3 \to SU(2), \qquad \vec{x} \mapsto \frac{\ell\,\mathbb{I} - 2\vec{x} \cdot \vec{t}}{\sqrt{\ell^2 + r^2}}, \tag{3.1.29}$$

whose image satisfies $G(\vec{x})^2 = H(\vec{x})$. This relation follows immediately from the explicit forms of the maps, but it also follows from the geometric meaning of G and H, which is illustrated in Fig. 3.2. For fixed $\vec{x}$, $G(\vec{x})$ and $H(\vec{x})$ are rotations about the same axis, and it follows from elementary geometry that G rotates by twice the rotation angle of H. As an aside we note that describing spherical geometry in terms of $\mathbb{R}^3$ via pull-back with H and G is analogous to describing hyperbolic geometry in terms of, respectively, the Poincaré and the Beltrami-Klein models. We will see these maps again in the setting of hyperbolic geometry in Chapter 4 and in the more general setting of the group manifold $\mathbb{H}^1_\lambda$ in Chapter 5

The maps G and H can be used to pull back the left-invariant 1-forms σ_j,

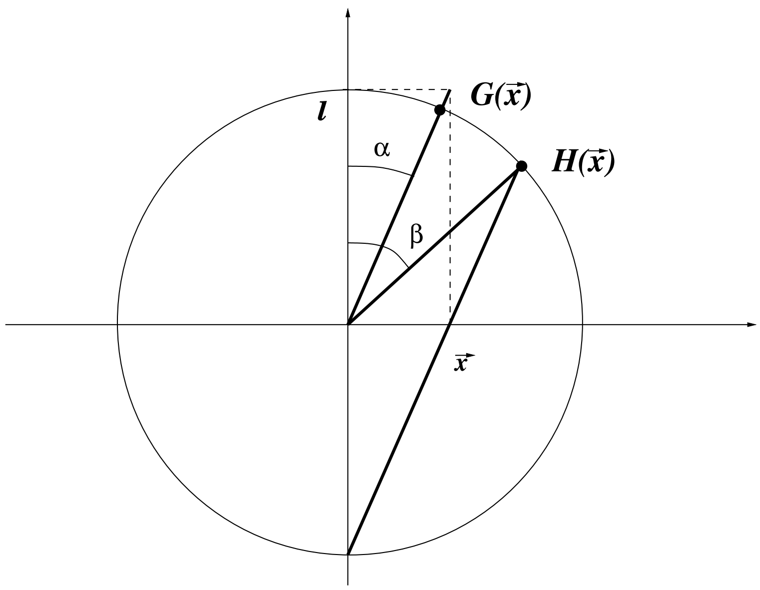

Figure 3.2: Geometry of the relation between the gnomonic and stereographic projections $G, H : \mathbb{R}^3 \to S^3$ defined in the main text. The relation $\beta = 2\alpha$ implies $H(\vec{x}) = G^2(\vec{x})$. Figure produced by B. Schroers

$j = 1, 2, 3$, from the 3-sphere to $\mathbb{R}^3$. The relation of the resulting frames on $\mathbb{R}^3$ to each other and to the standard frame (3.1.23) is important for the remainder of for this Chapter. We therefore collect the relevant results here. Defining the scale function

$$\Omega = \frac{\ell^2 + r^2}{4\ell}, \tag{3.1.30}$$

we introduce 1-forms θ_j, $j = 1, 2, 3$, on $\mathbb{R}^3$ via

$$H^{-1}dH = -\frac{1}{\Omega}\vec{\theta}\cdot\vec{t}, \quad \text{or} \quad \theta_j = -\Omega H^*\sigma_j. \tag{3.1.31}$$

Then we find

$$\vec{\theta}\cdot\vec{t} = \frac{1}{\ell^2 + r^2}\left(2(\vec{x}\cdot\vec{t})(\vec{x}\cdot d\vec{x}) + (\ell^2 - r^2)\vec{t}\cdot d\vec{x} + 2\ell(\vec{x}\times d\vec{x})\cdot\vec{t}\right) = G^{-1}(d\vec{x}\cdot\vec{t})G. \tag{3.1.32}$$

In other words, pulling back the 1-forms σ_1, σ_2 and σ_3 via H gives a frame which is related to the standard frame (3.1.23) by rotation with G (acting in its adjoint representation), a reflection in the origin and rescaling by Ω. For later use we note the explicit form of θ_3:

$$\theta_3 = \frac{1}{\ell^2 + r^2}\left(2(x_1x_3 - \ell x_2)dx_1 + 2(x_2x_3 + \ell x_1)dx_2 + (\ell^2 - r^2 + 2x_3^2)dx_3\right). \tag{3.1.33}$$

We now come to a result which relates the pullbacks of the Maurer-Cartan one-form, $h^{-1}dh$, with respect to the maps G and H.

Lemma 3.1
The pull-backs $G^{-1}dG$ and $H^{-1}dH$ of the Maurer-Cartan form on $SU(2)$ are related via

$$H^{-1}dH = G^{-1}dG + G^{-1}(G^{-1}dG)G, \tag{3.1.34}$$

with the inverse relation expressed as

$$G^{-1}dG = \frac{1}{2}H^{-1}dH + \star(d\Omega \wedge H^{-1}dH). \tag{3.1.35}$$

Here by $\star$ we mean the Hodge star on $SU(2)$ relating k-forms to $3-k$ forms through the volume form (3.1.22) .

Proof. The first formula (3.1.34) is an immediate consequence of $H = G^2$. To show the inverse relation, we use (3.1.34) to write (3.1.35) as

$$\star(2d\Omega \wedge (dGG^{-1} + G^{-1}dG)) = dGG^{-1} - G^{-1}dG. \tag{3.1.36}$$

Then computing

$$G^{-1}dG = -2\frac{\ell d\vec{x}\cdot\vec{t} + (\vec{x}\times d\vec{x})\cdot\vec{t}}{\ell^2 + r^2}, \tag{3.1.37}$$

we have

$$dGG^{-1} + G^{-1}dG = -4\frac{\ell d\vec{x}\cdot\vec{t}}{\ell^2 + r^2}, \quad dGG^{-1} - G^{-1}dG = 4\frac{(\vec{x}\times d\vec{x})\cdot\vec{t}}{\ell^2 + r^2}. \tag{3.1.38}$$

With $2d\Omega = \vec{x}\cdot d\vec{x}/\ell$ and

$$\star(\vec{x}\cdot d\vec{x} \wedge d\vec{x}\cdot\vec{t}) = d\vec{x}\times\vec{x}\cdot\vec{t}, \tag{3.1.39}$$

one deduces (3.1.36) and hence (3.1.35). □

3.1.3 Magnetic Dirac operators and their zero-modes

In the frame (3.1.20) for a 3-sphere of radius ℓ, a global gauge potential for the spin connection is

$$\Gamma_{S^3} = \frac{1}{2}h^{-1}dh, \tag{3.1.40}$$

Thus, the Dirac operator associated to the frame (3.1.20) is

$$\begin{aligned} \not{D}_{S^3} &= -\frac{4}{\ell}\left(t_1X_1 + t_2X_2 + t_3X_3\right) + \frac{3}{2\ell} \\ &= \frac{2i}{\ell}\begin{pmatrix} X_3 & X_- \\ X_+ & -X_3 \end{pmatrix} + \frac{3}{2\ell}. \end{aligned} \tag{3.1.41}$$

Minimal coupling to an abelian gauge potential $A = A_i\sigma^i$ gives

$$\begin{aligned} \not{D}_{S^3,A} &= -\frac{4}{\ell}\left(t_1(X_1 + iA_1) + t_2(X_2 + iA_2) + t_3(X_3 + iA_3)\right) + \frac{3}{2\ell} \\ &= \frac{2i}{\ell}\begin{pmatrix} (X_3 + iA_3) & (X_- + iA_-) \\ (X_+ + iA_+) & -(X_3 + iA_3) \end{pmatrix} + \frac{3}{2\ell}. \end{aligned} \tag{3.1.42}$$

The Dirac operator on $\mathbb{R}^3$ associated to the frame (3.1.23) and minimally coupled to an abelian gauge potential $A = A_1dx_1 + A_2dx_2 + A_3dx_3$ is

$$\not{D}_{\mathbb{R}^3,A} = -2t^j(\partial_j + iA_j), \tag{3.1.43}$$

where we used $\partial_i = \partial/\partial x^i$.

As we saw in the previous section, the frame (3.1.20) pulled back to $\mathbb{R}^3$ via H and the flat frame (3.1.23) are related by a rotation with G, a reflection and rescaling by (3.1.30). This leads to a simple relation between the zero-modes of the Dirac operators on S^3 and $\mathbb{R}^3$.

Lemma 3.2

If $\Psi : S^3 \to \mathbb{C}^2$ is a zero-mode of the Dirac operator (3.1.42) *on S^3 coupled to the $U(1)$ gauge field A, then*

$$\Psi_H = G\Omega^{-1}H^*\Psi \tag{3.1.44}$$

is a zero-mode of the Dirac operator $\not{D}_{\mathbb{R}^3,H^*A}$ on Euclidean 3-space coupled to the *connection H^*A.*

Proof. Since the spin connection in the frame (3.1.23) is manifestly zero, it follows from the equivariance of the Dirac operator under scaling and frame rotations that pulling back zero-modes of the Dirac operator on S^3 to $\mathbb{R}^3$ and applying the transformation $G\Omega^{-1}$ gives zero-modes of the Dirac operator on $\mathbb{R}^3$ in the flat frame (3.1.23). Here we check this explicitly. The pull-back of the spin connection is

$$H^*\Gamma_{S^3} = \frac{1}{2}H^{-1}dH. \tag{3.1.45}$$

Thus, using (3.1.34),

$$d + \frac{1}{2}H^{-1}dH = \Omega G^{-1}\left(d + \frac{1}{2}\left(GdG^{-1} + G^{-1}dG\right) + \Omega^{-1}d\Omega\right)\Omega^{-1}G. \tag{3.1.46}$$

Combining (3.1.38) and

$$\Omega^{-1}d\Omega = \frac{2\vec{x}\cdot d\vec{x}}{\ell^2 + r^2}, \tag{3.1.47}$$

one checks that

$$t^j\iota_{\partial_j}\left(\frac{1}{2}\left(GdG^{-1} + G^{-1}dG\right) + \Omega^{-1}d\Omega\right) = -\frac{2\vec{x}\cdot\vec{t}}{\ell^2 + r^2} + \frac{2\vec{x}\cdot\vec{t}}{\ell^2 + r^2} = 0. \tag{3.1.48}$$

Combining these results, one checks that the pull-back of the Dirac operator on S^3 coupled to the spin connection and the abelian connection A in the frame (3.1.23) is

$$-\frac{\ell}{4}H^*\not{D}_{S^3,A} = t^j\iota_{H^*X_j}(d + \frac{1}{2}H^{-1}dH + iH^*A) \tag{3.1.49}$$

$$= -\Omega G^{-1}t^j\iota_{\partial_j}G(d + \frac{1}{2}H^{-1}dH + iH^*A) \tag{3.1.50}$$

$$= -\Omega^2 G^{-1}t^j\iota_{\partial_j}(d + iH^*A)\Omega^{-1}G, \tag{3.1.51}$$

which implies the claimed relation between zero-modes of $\not{D}_{S^3,A}$ and $\not{D}_{\mathbb{R}^3,H^*A}$. Note that the components of the pull-back abelian connection relative to the frame

(3.1.23) are related to the components in the expansion $A = \vec{A} \cdot \vec{\sigma}$ via

$$H^* A = \vec{A}_H \cdot d\vec{x}, \qquad \vec{A}_H \cdot \vec{t} = -\frac{1}{\Omega} G^{-1} H^* \vec{A} \cdot \vec{t} G. \tag{3.1.52}$$

□

This result is essentially encapsulated in Theorem 23 in [14]. However, the version presented here is explicit in term of giving the maps G and H and an explicit form for the spinor.

This Lemma can be used to construct magnetic zero-modes on $\mathbb{R}^3$ from magnetic zero-modes on S^3. For the family constructed explicitly by Loss and Yau in [13], which we refer to here as linear, this was observed in [35] and elaborated in [36], where this family was obtained from eigenmodes of the Dirac operator on S^3. The corresponding argument for the family of vortex zero-modes is one of the main results of [10] and is included in Section 3.2.2 below.

We review the construction of the linear zero-modes briefly here because they will be needed later in this Chapter, expressed in our conventions for parametrising S^3 in terms of two complex variables. We define the functions

$$Y^j_{sm}(z_1, z_2) = C_{jms} \sum_k \frac{(-1)^{-k} z_1^{s-m+k} z_2^{j+m-k} \bar{z}_1^k \bar{z}_2^{j-s-k}}{(j+m-k)!k!(j-s-k)!(s-m+k)!}, \tag{3.1.53}$$

where C_{jms} is an overall normalisation constant

$$C_{jms} = (-1)^{j-s} \left((j+s)!(j-s)!(j+m)!(j-m)!\right)^{\frac{1}{2}}, \tag{3.1.54}$$

and

$$j \in \frac{1}{2}\mathbb{N}^0, \quad s, m = -j, -j+1, \ldots, j-1, j. \tag{3.1.55}$$

The summation index k runs over the values such that the factorials are well defined. These functions are orthonormal and satisfy

$$\Delta_{S^3} Y^j_{sm} = -j(j+1) Y^j_{sm}, \quad iZ_3 Y^j_{sm} = m Y^j_{sm}, \quad iX_3 Y^j_{sm} = s Y^j_{sm}. \tag{3.1.56}$$

as well as

$$iX_+Y^j_{sm} = \sqrt{j(j+1)-s(s+1)}Y^j_{s+1,m}, \quad iX_-Y^j_{sm} = \sqrt{j(j+1)-s(s-1)}Y^j_{s-1,m}. \tag{3.1.57}$$

Using the explicit expression for the Dirac operator given in (3.1.41), one deduces that

$$\Psi^{j+}_{sm} = \frac{1}{2j+1}\begin{pmatrix} \sqrt{j+s+1}Y^j_{sm} \\ \sqrt{j-s}Y^j_{s+1,m} \end{pmatrix} \tag{3.1.58}$$

is an eigenspinor of $\not{D}_{S^3}$ with eigenvalue

$$\lambda_+ = \frac{1}{\ell}\left(\frac{1}{2}+(2j+1)\right), \tag{3.1.59}$$

and that

$$\Psi^{j-}_{sm} = \frac{1}{2j+1}\begin{pmatrix} -\sqrt{j-s}Y^j_{sm} \\ \sqrt{j+s+1}Y^j_{s+1,m} \end{pmatrix} \tag{3.1.60}$$

is an eigenspinor of $\not{D}_{S^3}$ with eigenvalue

$$\lambda_- = \frac{1}{\ell}\left(\frac{1}{2}-(2j+1)\right), \tag{3.1.61}$$

with the restriction that $j > 0$. The degeneracy of each eigenvalue is $(2j+1)(2j+2)$.

We can now use a trick introduced by Loss and Yau [13] to obtain zero-modes of the gauged Dirac operator from general eigenmodes of the ungauged Dirac operator. Setting

$$\frac{1}{\ell}A_i = i\rho\frac{\Psi^\dagger t_i \Psi}{\Psi^\dagger\Psi}, \quad i = 1,2,3, \tag{3.1.62}$$

where $\Psi \neq 0$ and using $-4\Psi^\dagger t_i \Psi t_i = 2\Psi\Psi^\dagger - \Psi^\dagger\Psi\mathbb{I}$, one then has

$$\frac{4i}{\ell}\vec{A}\cdot\vec{t}\Psi = \rho\Psi. \tag{3.1.63}$$

With $A = A_i\sigma_i$, this implies

$$\not{D}_{S^3}\Psi = \rho\Psi \Leftrightarrow \not{D}_{S^3,A}\Psi = 0. \tag{3.1.64}$$

In general, one needs to check the validity of this result at the zeros of Ψ. We will do this later for our application of linear zero-modes. Assuming the zeros are dealt with, we can then apply Lemma 3.2 to obtain magnetic zero-modes on Euclidean 3-space of the form

$$\Psi_H = G\Omega^{-1}H^*\Psi^{j\pm}_{sm}. \tag{3.1.65}$$

3.2 Vortex equations and magnetic zero-modes

3.2.1 Vortex equations on S^3

We are now ready to introduce the 3-dimensional geometries which will lead us to the smooth vortex zero-modes and provide the link with Popov vortices on the 2-sphere. First, we define vortex configurations on the 3-sphere.

Definition 3.3

Let n be a positive integer, A be a 1-form on S^3 and $\Phi : S^3 \to \mathbb{C}$ be a complex-valued function. We say that the pair (Φ, A) is a vortex configuration on S^3 with vortex number $2n-2$ if the following conditions hold:

1. *Normalisation :*
$$A(X_3) = n - 1, \tag{3.2.1}$$
2. *Equivariance:*
$$\mathcal{L}_{X_3}A = 0, \quad i\mathcal{L}_{X_3}\Phi = (n-1)\Phi, \tag{3.2.2}$$
3. *Vortex equations:*
$$(d\Phi + iA\Phi) \wedge \sigma = 0, \qquad F_A = \frac{i}{2}(|\Phi|^2 - 1)\bar{\sigma} \wedge \sigma, \tag{3.2.3}$$
where $F_A = dA$.

The Equivariance conditions do not need to be included explicitly in the Definition, as stated in [15] they follow from the vortex equation, their contractions, and the normalisation condition. For the first Equivariance condition notice that $i_{X_3}dA = i_{X_3}F_A = 0$ follows from the second vortex equation, then Cartan's formula gives that $\mathcal{L}_{X_3}A = dA(X_3) = 0$. For the second equivariance condition contracting the first vortex equation with (X_3, X_-) gives $\mathcal{L}_{X_3}\Phi = -iA(X_3)\Phi$ which implies the equivariance condition.

The Definition is such that vortex configurations are mapped into vortex configurations by abelian gauge transformations of the form

$$(\Phi, A) \mapsto (e^{-i\alpha}\Phi, A + d\alpha), \qquad \alpha \in C^\infty(S^3), \quad X_3\alpha = 0. \tag{3.2.4}$$

Our vortex configurations on S^3 can be interpreted as an equivariant description of vortices on S^2 as follows. The normalisation condition (3.2.1) means that iA may be viewed as the connection 1-form on the total space $S^3/\mathbb{Z}_{2n-2}$ (Lens space) of a $U(1)$ bundle over S^2 of degree $2n-2$. Comparing with (3.1.18) and referring to [33] for details, the equivariance requirement (3.2.2) means that Φ is the equivariant form of a section of the associated line bundle (in this case the $(2n-2)$-th power of the hyperplane bundle). We will show in Section 3.3.1, Lemma 3.10, that the vortices on S^2 which are equivariantly described by our vortex configurations are in fact Popov vortices.

We note that contracting the first vortex equation with (X_+, X_-) gives

$$X_+\Phi + iA(X_+)\Phi = 0. \tag{3.2.5}$$

This equation will prove useful later in this Section and also in Section 3.3.1. However, we first show that the vortex equations on S^3 can be interpreted in terms of a flat non-abelian gauge field.

The following Theorem shows that any vortex configuration can be expressed in terms of the pull-back of the Maurer-Cartan form $h^{-1}dh$ on $SU(2)$ via a bundle map $U : S^3 \to S^3$ of the Hopf fibration covering a rational map $S^2 \to S^2$. Since the Maurer-Cartan form encodes the frame (3.1.20) of the round 3-sphere, its pull-back encodes the pull-back of the round metric with U. In that sense, this result is a 3-dimensional version of Baptista's interpretation of vortices as deformed 2-dimensional geometry. The Theorem is the three dimensional $\lambda = \lambda_0 = -1$ analogue of Theorem 2.3.

Theorem 3.4

A vortex configuration of degree $2n-2$ on S^3 determines a gauge potential for a flat

$su(2)$ connection on S^3 satisfying the normalisation condition

$$\mathcal{A}(X_3) = nt_3 \tag{3.2.6}$$

via the following expression:

$$\mathcal{A} = (A + \sigma_3)t_3 + \frac{1}{2}(\Phi\sigma t_- + \bar{\Phi}\bar{\sigma}t_+). \tag{3.2.7}$$

Conversely a flat $SU(2)$ connection $\mathcal{A}$ on $SU(2)$ satisfying (3.2.6) and

$$\mathcal{A}(X_-) = \beta t_3 + \Phi t_-, \tag{3.2.8}$$

for $\beta, \Phi : SU(2) \to \mathbb{C}$, determines a vortex configuration (A, Φ) through the expansion (3.2.7).

A gauge potential for a flat $su(2)$ connection on S^3 satisfying (3.2.6) and of the form (3.2.7) can be trivialised as $\mathcal{A} = U^{-1}dU$, where $U : S^3 \to S^3$ has degree n^2 and is a bundle map of the Hopf fibration, covering a rational map $R : S^2 \to S^2$ of degree n. Up to a $U(1)$ gauge transformation (3.2.4), one can choose the bundle map U to have the form

$$U : (z_1, z_2) \mapsto \frac{1}{\sqrt{|P_1|^2 + |P_2|^2}} \begin{pmatrix} P_1 & -\bar{P}_2 \\ P_2 & \bar{P}_1 \end{pmatrix}, \tag{3.2.9}$$

where P_1, P_2 are homogeneous polynomials of degree n with no common zeros

$$P_1 = a_0 z_1^n + a_1 z_1^{n-1} z_2 + \ldots + a_n z_2^n, \qquad P_2 = b_0 z_1^n + b_1 z_1^{n-1} z_2 + \ldots + b_n z_2^n, \tag{3.2.10}$$

and a_0, b_0, a_n, b_n all non-zero up to left multiplication of U by a constant $SU(2)$ matrix.

The vortex configuration (Φ, A) can be computed from the bundle map U via

$$U^*\sigma = \Phi\sigma, \qquad A = U^*\sigma_3 - \sigma_3, \tag{3.2.11}$$

and is given in terms of P_1, P_2 *by*

$$\Phi = \frac{P_1\partial_2 P_2 - P_2\partial_2 P_1}{z_1(|P_1|^2 + |P_2|^2)}, \tag{3.2.12}$$

and

$$A = (n-1)\sigma_3 + \frac{i}{2}X_- \ln(|P_1|^2 + |P_2|^2)\sigma - \frac{i}{2}X_+ \ln(|P_1|^2 + |P_2|^2)\bar{\sigma}. \tag{3.2.13}$$

The condition on a_0, b_0, a_n, b_n will turn out to be convenient in the discussion of Popov vortices in Section 3.3.2 and facilitates comparison with the treatment in [11].

Proof. Suppose (Φ, A) is a vortex configuration of degree $2n-2$. It is easy to check that, for a gauge potential of the form (3.2.7), the normalisation (3.2.1) implies (3.2.6). The flatness condition $d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0$ for a gauge potential of the form (3.2.7) is equivalent to

$$d(\Phi\sigma) + i(A + \sigma_3)\wedge\Phi\sigma = 0, \qquad dA = \frac{i}{2}(|\Phi|^2 - 1)\bar{\sigma}\wedge\sigma, \tag{3.2.14}$$

which, using (3.1.6), is equivalent to the vortex equations (3.2.3).

For the converse, expanding a flat $SU(2)$ connection, $\mathcal{A}$, in terms of the generators t_3, t_+, t_-, with the coefficients being linear combinations of the left invariant one-forms and imposing both the normalisation (3.2.6) and (3.2.8) results in the flat connection having the form (3.2.7) with Higgs field Φ and abelian gauge field

$$A = (n-1)\,\sigma^0 + \frac{1}{2}\left(\beta\sigma + \bar{\beta}\bar{\sigma}\right). \tag{3.2.15}$$

The vortex equations then follow from the flatness of $\mathcal{A}$. Finally the equivariance condition (3.2.2) for vortex configurations is equivalent to

$$\mathcal{L}_{X_3}\mathcal{A} = n[\mathcal{A}, t_3], \tag{3.2.16}$$

but this holds automatically for a flat gauge potential satisfying the normalisation

(3.2.6) since, for a flat gauge field,

$$\mathcal{L}_{X_3}\mathcal{A} = D_{\mathcal{A}}\mathcal{A}(X_3). \tag{3.2.17}$$

A flat and smooth $SU(2)$ gauge potential $\mathcal{A}$ on S^3 can always be globally trivialised in terms of a function $U : S^3 \to SU(2)$ as $\mathcal{A} = U^{-1}dU$. We now show that the vortex form (3.2.7) and the normalisation (3.2.6) forces the trivialising map to be a bundle map covering a rational map of degree n. The normalisation (3.2.6) requires that

$$X_3U = nUt_3, \quad \text{or} \quad U(he^{\gamma t_3}) = U(h)e^{n\gamma t_3}, \quad \gamma \in [0, 4\pi). \tag{3.2.18}$$

This equivariance condition has important topological consequences. It implies that the map $\pi \circ U$ is constant along fibres of the Hopf fibration and determines a map $S^2 \to S^2$; in the parametrisation of U in terms of two functions P_1, P_2 which do not vanish simultaneously as in (3.2.9) (but without assuming that P_1, P_2 are polynomials) this map is simply the quotient P_2/P_1. In terms of our stereographic coordinate z for S^2 and the section s in (3.1.17) we define

$$R = \pi \circ U \circ s, \tag{3.2.19}$$

and have the following commutative diagram (where we have not carefully distinguished between S^2 and our coordinate chart $\mathbb{C}$ for it):

$$\begin{array}{ccc} S^3 & \xrightarrow{U} & S^3 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^2 & \xrightarrow{R} & S^2. \end{array} \tag{3.2.20}$$

By virtue of (3.2.18), the map $\pi \circ U : S^3 \to S^2$ has Hopf number n^2: the pre-image of any point on S^2 is an n-fold cover of a circle which links with each of the n circles in the pre-image of another point exactly once. It follows that the map U has degree n^2 and the map R covered[2] by U has degree n.

Continuing in a parametrisation of U in terms of two functions P_1, P_2 but still

[2]Establishing this is a standard result about the Hopf invariant and is an exercise in [37].

not assuming that P_1, P_2 are polynomials, the condition (3.2.18) implies

$$2iX_3P_1 = nP_1, \quad 2iX_3P_2 = nP_2. \tag{3.2.21}$$

Since

$$U^{-1}dU = U^*\sigma_3 t_3 + \frac{1}{2}(U^*\sigma t_- + U^*\bar{\sigma} t_+), \tag{3.2.22}$$

we obtain a potential in the vortex gauge (3.2.7) if and only if

$$U^*\sigma = \Phi\sigma \tag{3.2.23}$$

for some function $\Phi : S^3 \to \mathbb{C}$. Using (3.1.15), we therefore need to show that

$$(U^*\sigma)(X_3) = 0, \qquad (U^*\sigma)(X_+) = 0, \tag{3.2.24}$$

are equivalent to U being a bundle map which covers a rational map. The first of these follows from (3.2.21), since

$$(U^*\sigma)(X_3) = \frac{2i}{|P_1|^2 + |P_2|^2}(P_1X_3P_2 - P_2X_3P_1). \tag{3.2.25}$$

To analyse the second condition, note that

$$(U^*\sigma)(X_+) = \frac{2i}{|P_1|^2 + |P_2|^2}(P_1X_+P_2 - P_2X_+P_1) = \frac{2i}{|P_1|^2 + |P_2|^2}P_1^2X_+\left(\frac{P_2}{P_1}\right), \tag{3.2.26}$$

with the last equality holding where $P_1 \neq 0$. As noted above, the ratio P_2/P_1 defines a function (section of the trivial bundle H^0) on S^2. According to the commutative diagram (3.1.19), $X_+(P_2/P_1) = 0$ means that the pull-back $R = s^*(P_2/P_1)$ is, in fact, a holomorphic function where it is defined. Thus for the second relation in (3.2.24) to hold R has to be a holomorphic map $S^2 \to S^2$ of degree n, which means it must be a rational map, as claimed.

These conditions are clearly satisfied when P_1 and P_2 are the homogeneous polynomials given in (3.2.9). In that case, the rational map is explicitly given by

$$R(z) = \frac{p_2(z)}{p_1(z)}, \tag{3.2.27}$$

where

$$p_1(z) = a_0 + a_1 z + \ldots + a_n z^n, \quad p_2(z) = b_0 + b_1 z + \ldots + b_n z^n. \tag{3.2.28}$$

In order for (3.2.27) to be a map of degree n we require at least one of a_n, b_n to be non-zero (so that the maximum of the degrees of p_1 and p_2 is n) and at least one of a_0, b_0 to be non-zero (so that we cannot reduce the degree by cancellation). We can then arrange for all of a_0, b_0, a_n, b_n to be non-zero by left-multiplying U with a constant $SU(2)$ matrix if necessary; this does not affect $\mathcal{A}$ and therefore leaves the vortex configuration unchanged.

Fixing U to be the trivialisation in terms of the polynomials P_1, P_2 in (3.2.9), we can define a new trivialisation

$$\tilde{U} = U e^{\alpha t_3}, \quad \alpha \in C^\infty(S^3), \qquad X_3 \alpha = 0. \tag{3.2.29}$$

This also satisfies (3.2.18), and leads to the same rational map R. The non-abelian gauge potential $\tilde{\mathcal{A}} = \tilde{U}^{-1} d\tilde{U}$ differs from $\mathcal{A} = U^{-1} dU$ by the gauge transformation (3.2.4), as claimed.

Continuing with P_1 and P_2 being homogeneous polynomials in z_1, z_2 of degree n, we obtain the claimed formula for the vortex field Φ from

$$\Phi = \frac{1}{2}(U^*\sigma)(X_-) = \frac{P_1 \partial_2 P_2 - P_2 \partial_2 P_1}{z_1(|P_1|^2 + |P_2|^2)}, \tag{3.2.30}$$

noting that

$$\frac{P_1 \partial_2 P_2 - P_2 \partial_2 P_1}{z_1} \tag{3.2.31}$$

is a homogeneous polynomial in z_1, z_2 of degree $2n-2$ and non-singular: the term of order z_2^{2n-1}, which could potentially cause a singularity when divided by z_1, vanishes.

The derivation of the expression for A is a straightforward calculation, which makes use of

$$(z_1 \partial_1 + z_2 \partial_2)(|P_1|^2 + |P_2|^2) = n(|P_1|^2 + |P_2|^2). \tag{3.2.32}$$

One finds

$$U^*\sigma_3(X_3) = n, \quad U^*\sigma_3(X_\pm) = \mp i X_\pm \ln(|P_1|^2 + |P_2|^2), \tag{3.2.33}$$

which, with (3.1.15), implies (3.2.13). □

In order to make contact with discussions in the literature related to the potential A we give an expression for A in terms of polar coordinates, for later use.

Lemma 3.5

Suppose (Φ, A) is a vortex configuration of vortex number $2n-2$ on S^3 and consider the modulus-argument parametrisation

$$\Phi = e^{M+i\chi}, \tag{3.2.34}$$

valid away from the (generically $2n-2$) zeros of Φ. Then the gauge potential A in (3.2.11) *can be expressed via the formula*

$$A = -\frac{\ell}{2} \star (\sigma_3 \wedge dM) - d\chi, \tag{3.2.35}$$

valid away from the zeros of Φ.

Proof. Observe that, away from the zeros of Φ, we can write (3.2.13) as

$$A = (n-1)\sigma_3 - \frac{i}{2} X_- \ln \bar{\Phi}\sigma + \frac{i}{2} X_+ \ln \Phi\bar{\sigma}. \tag{3.2.36}$$

Inserting the parametrisation (3.2.34) leads to

$$A - (n-1)\sigma_3 = \left(-\frac{i}{2}X_- M - \frac{1}{2}X_-\chi\right)\sigma + \left(\frac{i}{2}X_+ M - \frac{1}{2}X_+\chi\right)\bar{\sigma}. \tag{3.2.37}$$

With the Hodge-$\star$ relative to the orientation (3.1.22), we have

$$\star(\sigma_3 \wedge \sigma) = i\frac{2}{\ell}\sigma, \quad \star(\sigma_3 \wedge \bar{\sigma}) = -i\frac{2}{\ell}\bar{\sigma}, \tag{3.2.38}$$

so that

$$-\frac{i}{2}X_- M\sigma + \frac{i}{2}X_+ M\bar{\sigma} = -\frac{\ell}{2} \star (\sigma_3 \wedge dM), \tag{3.2.39}$$

where we have used that for any differentiable $f : S^3 \to \mathbb{C}$,

$$df = \frac{1}{2}X_- f\sigma + \frac{1}{2}X_+ f\bar{\sigma} + X_3 f\sigma_3, \tag{3.2.40}$$

Turning to the terms involving χ, using (3.2.40) and deducing from (3.2.2) that $X_3\chi = 1 - n$, we conclude that

$$d\chi = \frac{1}{2}X_-\chi\sigma + \frac{1}{2}X_+\chi\bar{\sigma} - (n-1)\sigma_3. \tag{3.2.41}$$

Combining (3.2.37) with (3.2.39) and (3.2.41) we arrive at the claimed expression for the gauge potential (3.2.11) in terms of the modulus and argument of the field Φ. □

3.2.2 Magnetic zero-modes from vortices

We are now ready to explain how to construct magnetic zero-modes of the Dirac operator on the 3-sphere and on Euclidean 3-space from vortex configurations on the 3-sphere. We define spinorial vortex zero-modes as follows.

Definition 3.6
A pair (Ψ, A) *of a spinor* Ψ *and a 1-form* A *on* S^3 *is said to be a vortex zero-mode of the Dirac equation on* S^3 *if*

$$\not{D}_{S^3,A}\Psi = 0, \qquad F_A = \frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh \Psi + \frac{1}{4}\sigma_1 \wedge \sigma_2, \tag{3.2.42}$$

where $\star$ *is the Hodge star operator on* S^3 *with respect to the metric* (3.1.21) *and orientation* (3.1.22).

Theorem 3.7
Suppose (Φ, A) *is a vortex configuration on* S^3*. Then the pair*

$$\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \qquad A' = A + \frac{3}{4}\sigma_3, \tag{3.2.43}$$

is a vortex zero-mode (Ψ, A') *on* S^3*.*

Proof. The spinor given in the Theorem is a zero-mode of the gauged Dirac equation if

$$\left(iX_3 - A'_3 + \frac{3}{4}\right)\Phi = 0 \quad \text{and} \quad X_+\Phi + iA'_+\Phi = 0. \tag{3.2.44}$$

However, $A'_3 = A'(X_3) = (n-1) + \frac{3}{4}$ so that the first of these equations follows from (3.2.2). The second follows from $A'(X_+) = A(X_+)$ and (3.2.5). Turning to the non-linear equation, we note that, for a spinor of the form given in the Theorem,

$$\frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh \Psi = \frac{4i}{\ell} |\Phi|^2 \star \left(-\frac{i}{2}\sigma_3 \right) = |\Phi|^2 \sigma_2 \wedge \sigma_1. \tag{3.2.45}$$

Moreover,

$$F_{A'} = F_A + \frac{3}{4}\sigma_2 \wedge \sigma_1 = \left(|\Phi|^2 - \frac{1}{4} \right) \sigma_2 \wedge \sigma_1, \tag{3.2.46}$$

so that the non-linear equation in the definition of a vortex zero-mode follows. □

We can pull back the vortex zero-modes of the Dirac equation on S^3 to $\mathbb{R}^3$ using Lemma 3.2, but we also need to understand how the non-linear equation behaves under this pull-back. It turns out that the resulting equations take their simplest form in vector notation for gauge potentials and their magnetic fields, i.e., when expanding a 1-form on $\mathbb{R}^3$ as $A = \vec{A} \cdot d\vec{x}$ and defining the magnetic field vector field via $dA = \frac{1}{2}\epsilon_{jkl} B_j dx_k \wedge dx_l$ or $\vec{B} = \nabla \times \vec{A}$.

The magnetic field corresponding to the inhomogeneous term is given by

$$\frac{1}{4} H^*(\sigma_1 \wedge \sigma_2) = \frac{4\ell^2}{(\ell^2 + r^2)^2} \star_{\mathbb{R}^3} \theta_3 = \frac{1}{2}\epsilon_{jkl} b_j dx_k \wedge dx_l, \tag{3.2.47}$$

$$\vec{b} = \frac{4\ell^2}{(\ell^2 + r^2)^3} \begin{pmatrix} 2(x_1 x_3 - \ell x_2) \\ 2(x_2 x_3 + \ell x_1) \\ \ell^2 - r^2 + 2x_3^2 \end{pmatrix}, \tag{3.2.48}$$

where we used (3.1.33). The integral lines of $\vec{b}$ are the fibres of the Hopf fibration (3.1.16); they are plotted in Fig. 3.3.

We claim that vortex zero-modes of the Dirac equation pull back to solutions of the following coupled equations in $\mathbb{R}^3$:

$$\not{D}_{\mathbb{R}^3, A} \Psi = 0, \qquad \vec{B} = -2i \Psi^\dagger \vec{t} \Psi + \vec{b}. \tag{3.2.49}$$

We state this result as follows.

Corollary 3.8

Any pair of homogeneous polynomials $P_1, P_2 : \mathbb{C}^2 \to \mathbb{C}$ of the same degree and

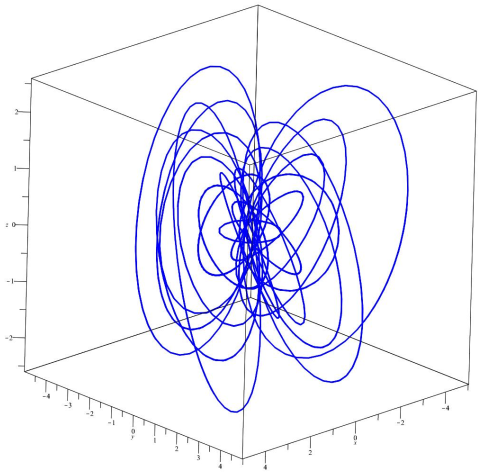

Figure 3.3: A plot of some of the integral curves of the background field $\vec{b}$ given in (3.2.47)

without common zeros uniquely determines a smooth and square-integrable magnetic zero-mode of the Dirac operator in Euclidean 3-space which satisfies the coupled equations (3.2.49).

Proof. Combining P_1 and P_2 into an $SU(2)$ matrix yields a vortex configuration (Φ, A) on S^3 according to the prescription of Theorem 3.4. Such a vortex configuration defines a vortex zero-mode Ψ of the Dirac operator on S^3 coupled to $A' = A + \frac{3}{4}\sigma_3$ according to Theorem 3.7. Implementing the conformal change to $\mathbb{R}^3$ according to Lemma 3.2 produces the magnetic zero-mode

$$\Psi_H = \Omega^{-1} G H^* \Psi = \Omega^{-1} G \begin{pmatrix} H^*\Phi \\ 0 \end{pmatrix} \tag{3.2.50}$$

on $\mathbb{R}^3$ of the Dirac operator coupled to $H^*A' = \vec{A'}_H \cdot d\vec{x}$.

We also need to understand the pull-back of the non-linear equation in (3.2.42). The quadratic term in the zero-mode Ψ on S^3 is

$$\frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh \Psi = i\epsilon_{jkl} \Psi^\dagger t_j \Psi \sigma_l \wedge \sigma_k, \tag{3.2.51}$$

and using (3.1.31) and (3.1.32) we deduce that pulling back with H yields

$$\begin{aligned} H^*\left(\frac{4i}{\ell}\star\Psi^\dagger h^{-1}dh\Psi\right) =& \frac{i}{\Omega^2}\epsilon_{jkl}H^*\Psi^\dagger G^{-1}t_jGH^*\Psi dx_l\wedge dx_k \\ =& -i\Psi_H^\dagger t_j\Psi_H\epsilon_{jkl}dx_k\wedge dx_l, \end{aligned} \tag{3.2.52}$$

where Ψ_H is related to Ψ as defined in (3.1.44). Finally expanding the pull-back of the field strength in the same coordinates

$$H^*F_{A'} = \frac{1}{2}\epsilon_{jkl}(B'_H)_jdx_k\wedge dx_l, \quad \text{so that} \quad \vec{B'}_H = \nabla\times\vec{A'}_H, \tag{3.2.53}$$

the non-linear equation in the definition (3.6) pulls back to

$$\vec{B'}_H = -2i\Psi_H^\dagger\vec{t}\Psi_H + \vec{b}, \tag{3.2.54}$$

as claimed.

The spinor Ψ_H in (3.2.50) and the gauge potential H^*A' are manifestly smooth, being the pull-back with smooth maps of smooth functions on S^3. As the pull-back of a smooth function on S^3, $H^*\Phi$ is bounded and has a (finite) limit as $r\to\infty$. It follows that

$$|\Psi_H^\dagger\Psi_H| \leq \frac{C}{\Omega^2}, \tag{3.2.55}$$

for some positive constant C, which ensures that Ψ_H is square-integrable with respect to the Euclidean measure (3.1.24). Since $|\Psi^\dagger 2i\vec{t}\Psi| = |\Psi^\dagger\Psi|^2$ for any spinor Ψ, it follows that the vector field $2i\Psi_H^\dagger\vec{t}\Psi_H$ is also square-integrable. The square-integrability of $\vec{B'}_H$ then follows from the square-integrability of $\vec{b}$ and the relation (3.2.54). $\square$

The coupled equations (3.2.49) have appeared in the literature in various contexts and deserve a few comments. There are various ways of stating these equations. Re-scaling the spinor by a factor of $\sqrt{2}$ leaves the linear equations unchanged, but changes the quadratic term in the non-linear equation into the spin density[3]

$$\vec{\Sigma} = i\Psi^\dagger \vec{t}\Psi. \tag{3.2.56}$$

Changing to the charge-conjugate spinor

$$\Psi^c = -2t_2\Psi, \tag{3.2.57}$$

turns our equations into the equivalent set of equations

$$\not{D}_{\mathbb{R}^3,-A}\Psi^c = 0, \qquad \vec{B} = 2i(\Psi^c)^\dagger \vec{t}\Psi^c + \vec{b}. \tag{3.2.58}$$

The equations (3.2.49) have been discussed in the literature as the dimensionally reduced Freund equations [31], while their charge-conjugates have appeared as the variational equations of a particular Dirac-Chern-Simons action [38].

The magnetic field on $\mathbb{R}^3$ obtained from the pair of complex polynomials P_1, P_2 can be written in terms of the Hopf map π and the maps H (3.1.27) and U (3.2.9) as

$$F = -(\pi \circ U \circ H)^*\mathcal{R}, \tag{3.2.59}$$

where $\mathcal{R}$ is the area form on the 2-sphere of unit radius (3.3.6), which will play an important role in the next section. In the same way the background field can be written as

$$F_b = \frac{1}{4}\left(\pi \circ H\right)^* \mathcal{R}. \tag{3.2.60}$$

These are examples of the magnetic fields introduced by Rañada in [3] and discussed more recently in [39]. It has interesting topological properties inherited from those of the map $U : S^3 \to S^3$, which, as explained after diagram 3.2.20, has topological degree n^2. As also explained there, $\pi \circ U : S^3 \to S^2$ has Hopf number n^2. As discussed in [4], this implies that the magnetic field (3.2.59) has linking number one and (magnetic) helicity n^2. We will return to give a more detailed discussion of magnetic fields of the Rañada form in Chapter 6.

In order to compare with solutions for (3.2.49) previously obtained in the lit-

[3]More comonnly the spin density would be written in terms of a vector of Pauli matrices, $\vec{\tau} = 2i\vec{t}$, as $\vec{\Sigma} = \frac{1}{2}\Psi^\dagger\vec{\tau}\Psi$.

erature, we also pull back the modulus-argument expression (3.2.35) to $\mathbb{R}^3$, to find

$$H^*A = -\frac{\ell}{2}H^*\left(\star(\sigma_3 \wedge dM)\right) - d(H^*\chi). \tag{3.2.61}$$

The operation

$$\star\sigma_3\wedge\ : \Lambda^1(S^3) \to \Lambda^1(S^3) \tag{3.2.62}$$

is linear; it annihilates σ_3 and acts as a complex structure on the cotangent space orthogonal to σ_3 by mapping

$$\star\sigma_3\wedge\ : \sigma \mapsto (2i/\ell)\sigma. \tag{3.2.63}$$

It pulls back to the map $(\theta_1 + i\theta_2) \mapsto (2i/\ell)(\theta_1 + i\theta_2)$, which we can write as

$$-\frac{2}{\ell}\star_{\mathbb{R}^3}\theta_3\wedge\ : \Lambda^1(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3). \tag{3.2.64}$$

Therefore

$$H^*A = -\star_{\mathbb{R}^3}\left(d(H^*M)\wedge\theta_3\right) - d(H^*\chi). \tag{3.2.65}$$

In [13], Loss and Yau showed that, for spinors on $\mathbb{R}^3$ whose spin density (3.2.56) has vanishing divergence, one can always find a gauge field A so that the given spinor is a zero-mode of the Dirac operator on $\mathbb{R}^3$ coupled to A. They gave an explicit formula, valid where the spinor does not vanish:

$$A_\Psi = -i\star\frac{d(\Psi^\dagger d\vec{x}\cdot\vec{t}\Psi)}{\Psi^\dagger\Psi} - \frac{\mathrm{Im}(\Psi^\dagger d\Psi)}{\Psi^\dagger\Psi}. \tag{3.2.66}$$

The relation to our expression (3.2.65) is as follows.

Lemma 3.9
Let (Φ, A) be a vortex configuration on S^3, and Ψ_H the corresponding zero-mode of the Dirac operator on $\mathbb{R}^3$ given in (3.2.50). Then the spin density of Ψ_H is divergenceless, and the corresponding Loss-Yau potential is given by

$$A_{\Psi_H} = H^*A + \frac{3}{4}H^*\sigma_3. \tag{3.2.67}$$

Proof. As shown above, the spinor Ψ_H (3.2.50) and the gauge potential $H^*A+\frac{3}{4}H^*\sigma_3$

satisfy the coupled equations (3.2.49). By virtue of the non-linear equation (3.2.54), the spin density for Ψ_H automatically has vanishing divergence. Using the expression (3.2.50) and the modulus-argument decomposition (3.2.34) pulled back to $\mathbb{R}^3$,i.e.,

$$H^*\Phi = e^{H^*M + iH^*\chi}, \tag{3.2.68}$$

one then computes

$$\begin{aligned} A_{\Psi_H} &= -\star\left(d(H^*M)\wedge\theta_3\right) - d(H^*\chi) \\ &\quad - (1,0)\left(i\Omega^2\star d\left(\frac{G^{-1}d\vec{x}\cdot\vec{t}G}{\Omega^2}\right) - iG^{-1}dG\right)\begin{pmatrix}1\\0\end{pmatrix} \end{aligned} \tag{3.2.69}$$

where all Hodge star operations now refer to $\mathbb{R}^3$. The first two terms combine to give the expression (3.2.65) for H^*A. The first term inside the expectation value can be re-written, using (3.1.32) and (3.1.31):

$$\begin{aligned} i\Omega^2\star d\left(\frac{G^{-1}d\vec{x}\cdot\vec{t}G}{\Omega^2}\right) &= -\Omega^2\star d\left(\frac{i}{\Omega}H^{-1}dH\right) \\ &= -iH^{-1}dH + i\star(d\Omega\wedge H^{-1}dH), \end{aligned} \tag{3.2.70}$$

where we used

$$\star d(H^{-1}dH) = \frac{1}{\Omega}H^{-1}dH. \tag{3.2.71}$$

Next we use the relation (3.1.35) to express $G^{-1}dG$ in terms of H and Ω, to deduce

$$i\Omega^2\star d\left(\frac{G^{-1}d\vec{x}\cdot\vec{t}G}{\Omega^2}\right) - iG^{-1}dG = -\frac{3i}{2}H^{-1}dH. \tag{3.2.72}$$

Then the observation

$$(1,0)\left(\frac{3i}{2}H^{-1}dH\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{3}{4}H^*\sigma_3, \tag{3.2.73}$$

completes the proof. □

The formula (3.2.66) is the starting point of several treatments in the literature of magnetic zero-modes, particularly in the papers [31, 32, 38] by Adam, Muratori and Nash (AMN). The AMN construction gives magnetic zero-modes in terms of solutions of the Liouville equation. However, by effectively pulling back local expressions for sections on S^2 to S^3 via the Hopf map it introduces additional singularities which we will discuss in more detail in Section 3.3.2.

3.2.3 Zero-mode combinatorics

It is natural to wonder if the linear magnetic zero-modes (3.1.65) and the vortex zero-modes (3.2.50) can be combined to produce new zero-modes. This is indeed possible when one picks $s = j$ in (3.1.65), and notes that, according to (3.1.53),

$$Y^j_{jm} = C_{jmj} z_1^{j-m} z_2^{j+m}, \quad j \in \frac{1}{2}\mathbb{N}^0, \tag{3.2.74}$$

so that a linear combination of such functions gives another homogeneous polynomial

$$P = A_0 z_1^{2j} + A_1 z_1^{2j-1} z_2 + \ldots + A_{2j} z_2^{2j}, \tag{3.2.75}$$

of degree $2j$. Since such a polynomial satisfies $iX_3 P = jP$ and $X_+ P = 0$, it is easy to check that one can combine it with a vortex configuration (Φ, A) of degree $2n - 2 \geq 0$ to get a solution

$$\Psi = G\Omega^{-1} H^* \begin{pmatrix} P\Phi \\ 0 \end{pmatrix}, \qquad H^* A' = H^* A + \left(j + \frac{3}{4}\right) H^* \sigma_3, \tag{3.2.76}$$

of the coupled Dirac equations (3.2.49) on $\mathbb{R}^3$. Physically, the inclusion of P in the spinor adds a multiple of the background field $\vec{b}$ to the solution.

There is an obvious mirror version of all our solutions in the anti-holomorphic world: for negative n and $s = -j$, one can write down vortex configurations $(\bar{\Phi}, -A)$ in terms of anti-holomorphic polynomials, and obtain corresponding Dirac zero-modes

$$\Psi = G\Omega^{-1} H^* \begin{pmatrix} 0 \\ \bar{P}\bar{\Phi} \end{pmatrix}, \tag{3.2.77}$$

of the Dirac operator coupled to $-H^* A - \left(j + \frac{3}{4}\right) H^* \sigma_3$. These are nothing but the charge-conjugates (3.2.57) of the holomorphic solutions (3.2.76).

3.3 Popov vortices on S^2 and Cartan connections

3.3.1 Popov vortices from vortices on S^3

We now turn to the link between our vortex equations on S^3 and the Popov vortex equations on S^2. The conventions of this section are those of Chapter 2 with $\lambda_0 = \lambda = -1$ but we repeat them here with a slight generalisation which takes account of the two-sphere having an arbitrary radius R rather than specialising to the case of radius 1.

In a stereographic coordinate z defined by projection from the south pole, the round metric of a 2-sphere of radius R is

$$ds^2 = \frac{4R^2 dz d\bar{z}}{(1+|z|^2)^2}, \tag{3.3.1}$$

and the complexified frame field $e = e_1 + ie_2$ is

$$e = \frac{2R}{1+|z|^2}dz, \quad \bar{e} = \frac{2R}{1+|z|^2}d\bar{z}. \tag{3.3.2}$$

In terms of this frame field, the structure equation (2.3.2) determines the spin connection 1-form Γ to be

$$\Gamma = i\frac{z d\bar{z} - \bar{z} dz}{1+|z|^2}. \tag{3.3.3}$$

The topology of the 2-sphere does not permit a globally defined frame, and one checks that the frame (3.3.2) is singular at $z = \infty$ (the south pole) by switching to $\zeta = 1/z$ [33] and noting that e behaves likes $\bar{\zeta}^2/|\zeta|^2 d\zeta$ near $\zeta = 0$; Γ, too, has a singularity at $z = \infty$. In our chart, the Riemann curvature form is

$$\mathcal{R} = d\Gamma, \tag{3.3.4}$$

and is related to the frame via the usual Gauss equation, Equation (2.3.5), rewritten here as

$$\mathcal{R} = K e_1 \wedge e_2 = \frac{i}{2R^2} e \wedge \bar{e}, \tag{3.3.5}$$

where $K = 1/R^2$ is the Gauss curvature which depends explicitly on the radius. Thus

$$\mathcal{R} = 2i\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}, \tag{3.3.6}$$

which integrates to 4π.

In [11], $R = \sqrt{2}$ is chosen and the Popov equations are expressed in terms of the associated Kähler form. However, as we shall see it is more natural to write them in terms of the Riemann curvature form. Specialising Definition 2.1 we have that a Popov vortex is defined on a principal $U(1)$ bundle of degree $2n-2$ over the 2-sphere. It is a pair (ϕ, a) of a connection a on this bundle and a section ϕ of the associated complex line bundle. With $a = a_z dz + a_{\bar{z}} d\bar{z}$ and $f = f_{z\bar{z}} dz \wedge d\bar{z} = da$, the vortex equations in [11, 12] are

$$\partial_{\bar{z}}\phi - i a_{\bar{z}}\phi = 0, \qquad F_a = (|\phi|^2 - 1)\mathcal{R}. \tag{3.3.7}$$

As also explained in [11], solutions are obtained from rational maps $R : S^2 \to S^2$ of degree n which, in our coordinate z, take the form (3.2.27). As explained in Chapter 2 Popov vortices are determined by

$$a = R^*\Gamma - \Gamma, \qquad R^*e = \phi e. \tag{3.3.8}$$

The second of these equations determines ϕ as

$$\phi = \frac{R'(1+|z|^2)}{1+|R|^2} = (p_2'p_1 - p_1'p_2)\frac{1+|z|^2}{|p_1|^2+|p_2|^2}\frac{\bar{p}_1}{p_1}, \tag{3.3.9}$$

which has singularities at the zeros of p_1 which we will discuss below (see also [11]).

We would like to relate the Popov equations and their solutions to vortices on the 3-sphere studied in Section 3.2.1. As reviewed in Section 3.1.1, the Hopf projection $S^3 \simeq SU(2) \to S^2$ in terms of complex coordinates (z_1, z_2) for $h \in SU(2)$ (see (3.1.3)) and the complex stereographic coordinate z on S^2 is $\pi : h \mapsto z_2/z_1$, and a local section of this bundle is given by (3.1.17).

Lemma 3.10

The pull-back of the vortex equations (3.2.3) *on* S^3 *via the section* s (3.1.17) *yields* *the Popov equations* (3.3.7) *up to a singular gauge transformation.*

Proof. With $U : SU(2) \to SU(2)$ defined in terms of polynomials P_1, P_2 as in (3.2.9) we define $R : S^2 \to S^2$ in our stereographic chart by

$$R = \pi \circ U \circ s. \tag{3.3.10}$$

This is the rational map already encountered in the proof of Theorem 3.4, see also the diagram (3.2.20); it is of the form (3.2.27). One checks that

$$\pi^*\Gamma = -\sigma_3 + i d \ln \frac{z_1}{\bar{z}_1}, \tag{3.3.11}$$

so that that the pull-back with (3.1.17) gives

$$s^*\sigma_3 = -\Gamma. \tag{3.3.12}$$

Pulling back (3.3.11) further with U

$$U^*\pi^*\Gamma = -U^*\sigma_3 + i d \ln \frac{P_1}{\bar{P}_1}, \tag{3.3.13}$$

and then with s,

$$s^*U^*\pi^*\Gamma = -s^*U^*\sigma_3 + i d \ln \frac{p_1}{\bar{p}_1} \tag{3.3.14}$$

we deduce from the definition of R that

$$R^*\Gamma = -s^*U^*\sigma_3 + i d \ln \frac{p_1}{\bar{p}_1}. \tag{3.3.15}$$

Thus we find that the pull-back of the 1-form $A = U^*\sigma_3 - \sigma_3$ is

$$s^*A = s^*U^*\sigma_3 - s^*\sigma_3 = -R^*\Gamma + \Gamma + i d \ln \frac{p_1}{\bar{p}_1}. \tag{3.3.16}$$

Also noting

$$s^*\Phi = \frac{p_1}{\bar{p}_1}\phi, \qquad s^*\sigma = \frac{i}{\lambda} e, \tag{3.3.17}$$

and combining this to pull-back the vortex equations on S^3 (3.2.3), we obtain

$$\begin{aligned}(d(s^*\Phi) + i(s^*A)(s^*\Phi)) \wedge e = 0,\\ ds^*A = \frac{i}{2\lambda^2}(|\phi|^2 - 1)\bar{e} \wedge e,\end{aligned} \tag{3.3.18}$$

or, equivalently,

$$\begin{aligned}\left(d\left(\frac{p_1}{\bar{p}_1}\phi\right) + i\left(-R^*\Gamma + \Gamma + id\ln\frac{p_1}{\bar{p}_1}\right)\left(\frac{p_1}{\bar{p}_1}\phi\right)\right) \wedge e = 0,\\ -d(R^*\Gamma) + d\Gamma = -(|\phi|^2 - 1)\mathcal{R}.\end{aligned} \tag{3.3.19}$$

Writing this in terms of the Popov connection a, we conclude that

$$\begin{aligned}(d\phi - ia\phi) \wedge e = 0,\\ -f = -(|\phi|^2 - 1)\mathcal{R},\end{aligned} \tag{3.3.20}$$

which is equivalent to the Popov equations (3.3.7). □

3.3.2 Geometrical interpretation and singularities

It is implicit in our summary, particularly in equation (3.3.8), that Popov vortices can be interpreted purely geometrically. A metric viewpoint was emphasised and discussed in the more general context of vortex equation on a Riemann surface with a Kähler metric in [2]. This viewpoint was summarised in Chapter 2. There the singularities of the curvature in degenerate frame ϕe were discussed as well as how the addition of delta function singularities to this curvature, Equation (2.5.9), ensures that it still satisfies the Gauss-Bonnet formula.

It is interesting to note that in the metric interpretation, the zeros of the Higgs field lead to singularities whereas the actual singularities of the Higgs field do not appear to play a special role. To understand the geometric interpretation of the singularities of the Higgs field, we need to consider the frame field defined by it.

The complexified frame field

$$\phi e = 2\lambda \frac{p_2' p_1 - p_1' p_2}{|p_1|^2 + |p_2|^2} \frac{\bar{p}_1}{p_1}\, dz \tag{3.3.21}$$

has singularities at each the zeros of p_1, i.e., at each of the pre-images of the singularity of the frame e under the map R. If q is a zero of p_1, the behaviour near q is

$$\phi e \sim A \frac{\bar{z} - \bar{q}}{z - q} dz, \tag{3.3.22}$$

for some constant A. Near $z = \infty$, we use again $\zeta = 1/z$ to write the leading term as

$$\phi e \sim B \frac{dz}{z^2} = -B d\zeta, \quad B \text{ constant}, \tag{3.3.23}$$

which is smooth. It follows that the winding number of the frame field is localised at the zeros of p_1, with each zero (counted with multiplicity) contributing a winding of 4π.

This interpretation is gauge dependent. Using (3.3.17), we find that the frame field

$$s^*(\Phi\sigma) = 2i \left(\frac{p_2' p_1 - p_1' p_2}{|p_1|^2 + |p_2|^2} \right) dz \tag{3.3.24}$$

has no singularities for finite z, but has a singularity at $z = \infty$, where it behaves like

$$s^*(\Phi\sigma) \sim C \left(\frac{z}{|z|} \right)^{2n} \frac{dz}{z^2} = -C \left(\frac{\bar{\zeta}}{|\zeta|} \right)^{2n} d\zeta \tag{3.3.25}$$

for yet another constant C. In this gauge, the full phase rotation of $4\pi n$ is concentrated at $z = \infty$.

Our discussion shows that any description of the magnetic zero-modes in terms of the Popov vortex fields invariably has singularities since the Popov vortex is a section of and a connection on a non-trivial bundle, neither of which permits a globally smooth expression. This also applies to the expressions derived in [32, 38], which, in our terminology, express the magnetic zero-modes in terms of the modulus and phase of a scalar Popov vortex field (whose modulus obeys a Liouville equation). While one can shift the location of the singularities with gauge transformations, one cannot remove them on S^2.

3.3.3 Gauge potentials for Cartan connections

We saw in Chapter 2 that the Popov vortex equations can be described in terms of a gauge potential for a particular flat $SU(2)$ connection, Theorem 2.3. In particular it is the pullback with the rational map R of the flat connection

$$\hat{A} = -\Gamma t_3 + \frac{i}{2R}\left(et_- - \bar{e}t_+\right), \tag{3.3.26}$$

which encodes the round geometry of S^2. Explicitly this vortex gauge potential is

$$R^*\hat{A} = -\left(a + \Gamma\right)t_3 + \frac{i}{2R}\left(\phi e t_- - \bar{\phi}\bar{e}t_+\right). \tag{3.3.27}$$

The gauge potential $R^*\hat{A}$ inherits singularities from the singularities of ϕe discussed earlier. In order to treat this more carefully, we use the notion of a principal divisor $D = \sum n_j q_j$ of degree n on S^2. Given such a divisor we construct a bundle over $S^2 \setminus \{q_j\}$ by removing the union of the fibres over the q_j from $SU(2)$, obtaining the total space

$$P_D = SU(2) \setminus \bigcup_j \pi^{-1}(q_j). \tag{3.3.28}$$

For a homogeneous polynomial P of degree n in z_1, z_2, let D be the divisor of zeros of the associated inhomogeneous polynomial p (so $P(z_1, z_2) = z_1^n p\left(\frac{z_2}{z_1}\right)$). Then we can define the map

$$r_P : P_D \to SU(2), \quad r_P = \begin{pmatrix} \frac{\bar{P}}{|P|} & 0 \\ 0 & \frac{P}{|P|} \end{pmatrix}, \tag{3.3.29}$$

and the pull-back

$$r_p = s^* r_P : S^2 \setminus \{q_j\} \to SU(2). \tag{3.3.30}$$

It has the form

$$r_p = \begin{pmatrix} \frac{\bar{p}}{|p|} & 0 \\ 0 & \frac{p}{|p|} \end{pmatrix}. \tag{3.3.31}$$

For later use we note the behaviour of this matrix under fibre rotations. Identifying

h with (z_1, z_2) as in (3.1.3), we have

$$r_P(he^{\frac{\gamma}{n}t_3}) = e^{-\gamma t_3} r_P(h), \quad \gamma \in [0, 4\pi)\,. \tag{3.3.32}$$

Lemma 3.11

With s defined as in (3.1.17), the gauge potential for the Cartan connection of the 2-sphere is trivialised by s:

$$\hat{A} = s^{-1}ds. \tag{3.3.33}$$

Moreover, if U is the bundle map (3.2.9) covering the rational map $R = p_2/p_1$, the gauge potential R^*A for the deformed Cartan geometry and the pull-back via s of $\mathcal{A} = U^{-1}dU$ are related through the singular gauge transformation r_{p_1}:

$$R^*\hat{A} = r_{p_1}^{-1} s^* \left(\mathcal{A}\right) r_{p_1} + r_{p_1}^{-1} dr_{p_1}. \tag{3.3.34}$$

Proof. The formula (3.3.33) follows by an elementary calculation and comparison with the definition of e and Γ in terms of z in (3.3.2) and (3.3.3). With the map $U : S^3 \to S^3$ defined in terms of polynomials P_1, P_2 as in (3.2.9), and the map $R : S^2 \to S^2$ defined as in (3.3.10), one checks that,

$$U \circ s = \frac{1}{\sqrt{|p_1|^2 + |p_2|^2}} \begin{pmatrix} p_1 & -\bar{p}_2 \\ p_2 & \bar{p}_1 \end{pmatrix}, \tag{3.3.35}$$

and so, choosing the polynomial P_1 used in the definition of U (3.2.9),

$$s \circ R = (U \circ s) r_{p_1}. \tag{3.3.36}$$

It follows that

$$(s \circ R)^{-1} d(s \circ R) = r_{p_1}^{-1} s^* \left(U^{-1}dU\right) r_{p_1} + r_{p_1}^{-1} dr_{p_1}. \tag{3.3.37}$$

Since $\mathcal{A} = U^{-1}dU$ and

$$R^*\hat{A} = (s \circ R)^{-1} d(s \circ R), \tag{3.3.38}$$

the claim follows. $\square$

While the 1-form $\mathcal{A} = U^{-1}dU$ is manifestly smooth on S^3, its pull-back with s is not. The map $U \circ s$, (3.3.35), has a singularity of the form $z^n/|z|^n$ at $z = \infty$, as one would expect since the pull-backs s^*P_1 and s^*P_2 are local expression for sections of line bundles of degree n over S^2 [33]. It follows that the pull-back $s^*\mathcal{A}$ is singular at $z = \infty$, with the singularity already exhibited at (3.3.25).

3.3.4 Cartan geometry

Our description of the geometry of the 2-sphere and its pull-back via the rational map R in terms of $su(2)$ gauge potentials has been entirely local so far. It is time to address the global geometrical structure behind these gauge potentials. We will specify the bundles and the connections for which (3.3.26) and (3.3.27) are local gauge potentials in the language of Cartan geometry, but refer the reader to the textbook [29] and particularly to the PhD thesis [30] for general definitions and facts about Cartan geometry.

Cartan connections describe the geometry of manifolds modelled on homogeneous spaces G/H in terms of a connection on a principal G-bundle Q over this manifold. In order to recover the geometry of a manifold from a Cartan connection one needs an additional structure, namely a section of an associated G/H bundle which is transverse to the connection or, equivalently (as explained in [30]), a principal H subbundle P of Q which is transverse to the connection.

Here, we are interested in the case $G = SU(2), H = U(1)$ and $G/H = S^2$, and only consider flat Cartan connections. However, we will need to extend the usual framework of Cartan connections to deal with singularities. Consider a divisor D of degree n on S^2, and define the quotient

$$P_{D,n} = P_D/\mathbb{Z}_n, \tag{3.3.39}$$

where P_D is as defined in (3.3.28) and we think of $\mathbb{Z}_n$ as the subgroup generated by $e^{\frac{4\pi}{n}t_3}$, acting from the right on $SU(2)$. This is a $U(1)$ bundle over $S^2 \setminus \{q_j\}$ with the projection provided by the usual Hopf map π (3.1.16). It is a Lens space with n circles removed.

In order to construct the required principal $SU(2)$ bundle, we define the $SU(2)$-

bundle associated to P_D via a $U(1)$ action on $SU(2)$:

$$Q_D = \{(h, g) \in P_D \times SU(2)\}/\sim, \tag{3.3.40}$$

where $\sim$ is the equivalence relation

$$(h, g) \sim \left(he^{\frac{\gamma}{n}t_3}, ge^{\gamma t_3}\right), \quad \gamma \in [0, 4\pi). \tag{3.3.41}$$

This is an $SU(2)$ bundle over $S^2 \setminus \{q_j\}$ with projection

$$\Pi : Q_D \to S^2 \setminus \{q_j\}, \quad \Pi((h, g)) = \pi(h). \tag{3.3.42}$$

To make this into a principal $SU(2)$ bundle, we pick a homogeneous polynomial P of degree n and consider the map r_P in (3.3.29). Then

$$\tilde{g} = gr_P \tag{3.3.43}$$

is well-defined on Q_D, by (3.3.32), and we define the $SU(2)$ right-action as

$$u : (h, g) \mapsto (h, gr_P u). \tag{3.3.44}$$

Sections are constructed from maps satisfying an equivariance condition

$$U : P_D \to SU(2), \quad U\left(he^{\frac{\gamma}{n}t_3}\right) = U(h)\left(e^{\gamma t_3}\right), \quad \gamma \in [0, 4\pi). \tag{3.3.45}$$

This ensures that, for any U satisfying this condition,

$$S_U : S^2 \setminus \{q_j\} \to Q_D, \quad n \mapsto (h, U(h)), \qquad \pi(h) = n, \tag{3.3.46}$$

is a well-defined section.

The Maurer-Cartan form $g^{-1}dg$ is not well-defined on Q_D since it is not right-invariant. However, for any P, the element $\tilde{g}$ defined in (3.3.43) is, and so

$$\omega = \tilde{g}^{-1}d\tilde{g} \tag{3.3.47}$$

is well-defined and satisfies the equivariance condition for a connection 1-form. It is the Cartan connection which we are looking for. The map U in (3.2.9) satisfies (3.3.45). Picking $P = P_1$ and pulling back ω to our stereographic coordinate chart via S_U leads to the gauge potential

$$(S_U)^*\omega = s^*((Ur_{P_1})^{-1}d(Ur_{P_1}) = (s \circ R)^{-1}d(s \circ R), \tag{3.3.48}$$

which, according to (3.3.38) and (3.3.27), indeed captures the geometry induced by the Popov vortices.

To end this section, we also exhibit the second way in which one recovers geometry from a Cartan connection. As mentioned earlier, this requires a transverse section of the $SU(2)/U(1) \simeq S^2$ bundle associated to the principal $SU(2)$ bundle Q_D. In the trivialisation via (3.3.46), this section (often called the Higgs field in the physics literature on Cartan connections) is simply the constant map

$$\varphi : \mathbb{C} \to S^2, \quad z \mapsto t_3, \tag{3.3.49}$$

where we think of t_3 as an element of unit sphere inside $su(2)$. The geometry is recovered from the covariant derivative

$$D_{R^*\hat{A}}\varphi = [R^*\hat{A}, t_3], \tag{3.3.50}$$

which extracts the frame ϕe from the gauge potential $R^*\hat{A}$. If we apply the gauge transformation $s^*(Ur_{P_1})$, the gauge potential vanishes in the new gauge, but the transverse section is now

$$\tilde{\varphi} : z \mapsto Ut_3U^{-1}, \tag{3.3.51}$$

which, up to stereographic projection, is our rational map R. In other words, the rational map which solves the Popov vortex equations is the 'transverse Higgs field' of Cartan geometry in a particular gauge. The geometry is still recovered via the covariant derivative, but since the gauge potential vanishes now, this is simply the exterior derivative $d\tilde{\varphi}$ which, modulo stereographic projection, indeed reproduces the formula for the frame ϕe in terms of the derivative of R.

Chapter 4

The Lorentzian story

This Chapter is a close adaptation of the paper [15] and is the Lorentzian analogue of both the paper [10] and Chapter 3. Here the relationship between hyperbolic vortices, the $\lambda_0 = \lambda = 1$ vortices from Chapter 2, and vortex configurations on $SU(1,1)$ is explored before using these to construct magnetic Dirac modes on both $SU(1,1)$ and $\mathbb{R}^{1,2}$. This Chapter will proceed in the opposite direction to Chapter 3 and starts with a discussion of hyperbolic vortices and their relationship to vortex configurations on $SU(1,1)$.

Figure 4.1 is the Lorentzian version of Figure 3.1 and provides an overview of all the equations which are considered in this Chapter.

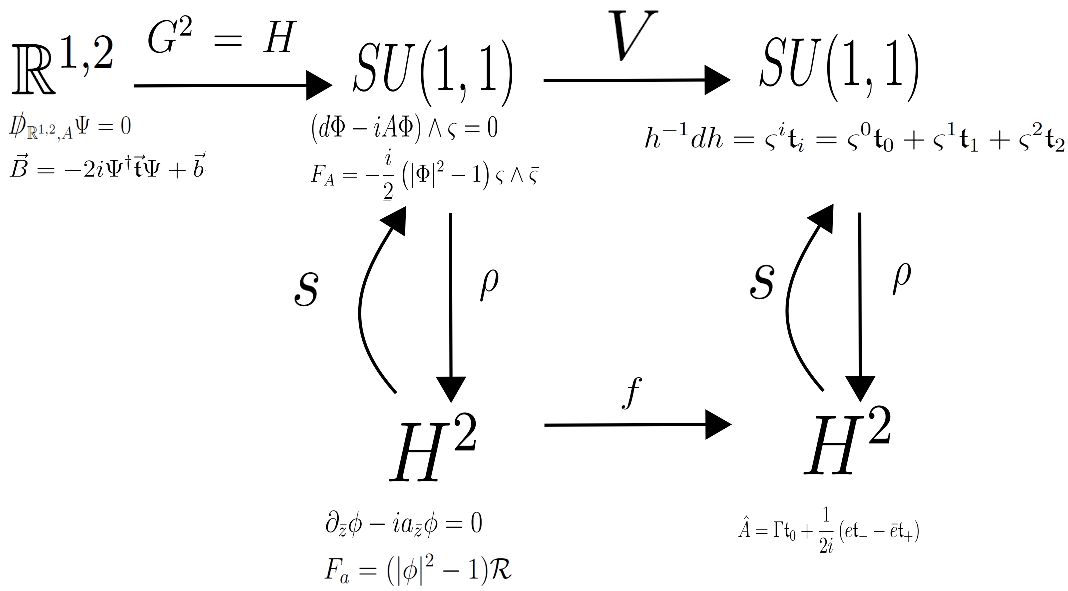

Figure 4.1: This Figure summarises all the spaces and equations considered here as well as the maps which relate them.

4.1 Notation specific to this Chapter

Before getting started it needs to be noted that the conventions of [15], which we adopt in this Chapter, differ from those used in Chapter 2. In particular the $SU(1,1)$ generators are taken differently to Equation (2.2.11) with the t_0 generator differing by a minus sign. As mentioned in a footnote in Chapter 2 this difference is down to taking the Lie algebra of $su(1,1)$ to have inverse metric $(g^{ij}) = \mathrm{diag}(-1,1,1)$ in Chapter 2 while in this chapter we pick the opposite signature $(g^{ij}) = \mathrm{diag}(1,-1,-1)$. This will have a knock on effect on the sign of the left-invariant one-form σ^0. We choose to work with this reflection of the $su(1,1)$ Lie algebra as it results in the Killing form on the Lie algebra lining up with its metric which simplifies many of the computations.

To try and make this distinction clear in this Chapter we will use $\mathfrak{t}_i$ for the generators, ς^i for the left-invariant one-forms and $\mathfrak{X}_i$ for the left-invariant vector fields. In the interest of being completely clear this means that the generators are

$$\mathfrak{t}_0 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathfrak{t}_1 = -\frac{i}{2}\begin{pmatrix} 0 & -\lambda \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{t}_2 = \frac{1}{2}\begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}, \tag{4.1.1}$$

and they obey the commutation relations

$$[\mathfrak{t}_i, \mathfrak{t}_j] = \varepsilon_{ij}{}^{k}\mathfrak{t}_k. \tag{4.1.2}$$

To convert any of the results of this Chapter to the conventions of the rest of this thesis the following substitutions need to be made

$$\mathfrak{t}_0 \mapsto -t_0, \;\; \varsigma^0 \mapsto -\sigma^0, \;\; \mathfrak{X}_0 \mapsto -X_0, \tag{4.1.3}$$

with the others being the same in both sets of conventions.

4.2 Hyperbolic vortices and Cartan geometry

4.2.1 Hyperbolic vortices and holomorphic maps

We now turn to the integrable $\lambda = \lambda_0 = 1$ vortex equations. First order vortex equations on the Poincaré disk model of hyperbolic space have been studied extensively, with many of the details summarised in [9]. Solutions can be obtained from $SO(3)$ invariant instantons on $S^2 \times H^2$, [17], and expressed in terms of holomorphic mappings of the disk. These were discussed in Chapter 2 and we review some of the specific details here. The reader should be warned that our disk has radius 1 rather than $\sqrt{2}$ as chosen in [9]. We also continue the convention of Chapters 2 and 3 and write the equations in terms of the Riemann curvature form rather than the Kähler form for the disk.

We first summarise our notation for the geometry of the Poincaré disk model, which we write as

$$H^2 = \{z \in \mathbb{C} | |z|^2 < 1\}, \tag{4.2.1}$$

with metric

$$ds^2 = \frac{4dzd\bar{z}}{(1-|z|^2)^2}. \tag{4.2.2}$$

The complexified orthonormal frame field for this metric is

$$e = \frac{2}{1-|z|^2}dz, \quad \bar{e} = \frac{2}{1-|z|^2}d\bar{z}, \tag{4.2.3}$$

with the Kähler form being

$$\omega = \frac{i}{2} e \wedge \bar{e} = 2i \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}. \tag{4.2.4}$$

The structure equation (2.3.2) determine the spin connection 1-form Γ to be

$$\Gamma = i\frac{\bar{z}dz - zd\bar{z}}{1-|z|^2}. \tag{4.2.5}$$

These are the $\lambda = 1$ case of the local geometry conventions from Chapter 2.

In contrast to the discussion of Popov vortices in Chapter 3 the frame fields are global and the spin connection has no singularities.

The Riemann curvature form is given by

$$\mathcal{R} = d\Gamma, \tag{4.2.6}$$

and is related to the Gauss curvature, K, and the Kähler form, equation (4.2.4), through the Gauss equation,

$$\mathcal{R} = K\omega, \tag{4.2.7}$$

with $K = -1$ for H^2.

Specialising Definition 2.1 a hyperbolic vortex is a pair (ϕ, a) where a is a connection on a principal $U(1)$ bundle over H^2, which is necessarily trivial, and ϕ is a smooth section of the associated complex line bundle. Taking $a = a_z dz + a_{\bar{z}} d\bar{z}$ and $F_a = da$, the vortex equations are

$$\partial_{\bar{z}}\phi - i a_{\bar{z}}\phi = 0, \quad F_a = (|\phi|^2 - 1)\mathcal{R}. \tag{4.2.8}$$

The first of these requires that ϕ be holomorphic with respect to the connection a. In the form presented here, these equations look the same as the Popov vortex equations from Chapter 3 since the sign difference due to the Gauss curvature is absorbed into the Riemann curvature two-form.

Solutions to the hyperbolic vortex equations can be constructed from holomorphic functions $f : H^2 \to H^2$ in the following way. Given the complex frame and spin connection, e, Γ on H^2, we pull them back via f and define the Higgs field and gauge potential through

$$\phi e = f^* e, \quad a = f^*\Gamma - \Gamma, \tag{4.2.9}$$

so that we have an explicit expression for ϕ in terms of f:

$$\phi = \frac{1 - |z|^2}{1 - |f|^2} f'. \tag{4.2.10}$$

This is explained fully in Chapter 2 (see Equation (2.4.8)).

As discussed before the pulled back frame f^*e degenerates at the zeros of ϕ. This corresponds to the vortex positions becoming conical singularities of the Baptista metric (2.5.2), with an angular excess related to the charge of the vortex, see [1] and Chapter 2. We saw in Chapter 2 that a consequence of the frame being degenerate is that its spin connection, $\tilde{\Gamma}$, has singularities. However, $f^*\Gamma$ is the pullback of a smooth spin connection with a smooth map, so is in particular non-singular. The difference between the pulled back spin connection and the spin connection of the degenerate frame is due to singularities at the zeros of ϕ. This is encapsulated in (2.5.9) with $\tilde{\mathcal{R}}$ being equal to $f^*\mathcal{R} = \mathcal{R} + F_a$, up to the addition of delta function singularities at the zeros z_j of ϕ.

One can view Riemann surfaces of genus $g > 1$ as the quotient of H^2 by the action of a Fuchsian group $\Gamma < SU(1,1)$. Vortices on such Riemann surfaces can therefore be constructed from vortices on H^2 that are invariant under the action of the desired Fuchsian group. In practice, this is not easy. A vortex solution on the Bolza surface (genus 2) is presented in [28]. While these vortices have an infinite number of zeros of the Higgs field on H^2 they have a finite number of zeros within the principal domain of the Fuchsian group.

In the construction of vortices from holomorphic maps between compact Riemann surface via (4.2.9), the Gauss-Bonnet theorem imposes a constraint on the genus of the surfaces and the vortex number, see [2]. The negative contribution from the singularities in the curvature $\tilde{\mathcal{R}}$, (2.5.9), at the zeros of ϕ plays a key role here, and provides a no-go theorem in some cases.

We now specialise to the case of solutions of the vortex equations on H^2 which satisfy the boundary condition $|\phi| \to 1$ as $|z| \to 1$. As explained in [9, 25], this boundary condition is required for the vortex to have finite energy. It also means that ϕ has a finite vortex number associated to it, which is the number of zeros counted with multiplicity. This is analogous to the degree of the line bundle for the case of Popov vortices discussed in Chapter 2.

As was first observed in [17], solutions of the hyperbolic vortex equations on H^2 which satisfy the boundary condition are obtained from bounded holomorphic functions $f : H^2 \to H^2$ which can be expressed as a finite Blaschke product when working in the Poincaré disk model. For a $(N-1)$-vortex solution, the Blaschke product can be written as the ratio

$$f = \frac{f_2}{f_1} \tag{4.2.11}$$

of the two holomorphic functions

$$f_1(z) = \prod_{k=1}^{N} (1 - \bar{c}_k z)\,, \quad f_2(z) = \prod_{k=1}^{N} (z - c_k)\,, \tag{4.2.12}$$

where $c_k \in H^2$, $k = 1, \ldots, N$. As the zeros of f_2 are in the disk of radius 1 the zeros of f_1, at $1/\bar{c}_k$, are not. Thus f has zeros but no poles within the disk. Note that $|f(z)| = 1$ when $|z| = 1$, i.e., on the boundary of the disk and that, by the maximum principle, $|f(z)| < 1$ when $|z| < 1$, so that f really is a holomorphic mapping of the disk model.

This way of writing the holomorphic function will prove useful later when we introduce and discuss vortex configurations on $SU(1,1)$, as will the observation about the lack of poles. The pullback of the holomorphic frame field has the explicit form

$$f^*e = \phi e = 2i\frac{f_2' f_1 - f_1' f_2}{|f_1|^2 - |f_2|^2}\frac{\bar{f}_1}{f_1}dz. \tag{4.2.13}$$

This is a manifestly smooth one-form which vanishes at the zeros of ϕ, thus illustrating our earlier remarks about the pullback frame.

4.2.2 Interlude on $SU(1,1)$

Before we interpret hyperbolic vortices in terms of Cartan geometry we state our conventions for the pseudo-unitary group $SU(1,1)$ in this Chapter. It is defined as the subgroup of $SL(2,\mathbb{C})$ whose elements h satisfy

$$h\tau_3 h^\dagger = \tau_3, \tag{4.2.14}$$

where τ_3 is the third Pauli matrix. Its Lie algebra $su(1,1)$ is defined as the set of complex traceless matrices satisfying

$$g^\dagger = -\tau_3 g \tau_3. \tag{4.2.15}$$

This forces the diagonal elements to be purely imaginary and the off-diagonal elements to be mutually complex-conjugate. We work with the generators given in (4.1.1) which obey the commutation relation

$$[\mathfrak{t}_i, \mathfrak{t}_j] = \varepsilon_{ij}{}^k \mathfrak{t}_k. \tag{4.2.16}$$

We also frequently use

$$\mathfrak{t}_+ = \mathfrak{t}_1 + i\mathfrak{t}_2, \quad \mathfrak{t}_- = \mathfrak{t}_1 - i\mathfrak{t}_2, \tag{4.2.17}$$

which have commutators

$$[\mathfrak{t}_+, \mathfrak{t}_-] = -2i\mathfrak{t}_0, \quad [\mathfrak{t}_0, \mathfrak{t}_\pm] = \pm i\mathfrak{t}_\pm, \tag{4.2.18}$$

showing that $\mathfrak{t}_0$ acts as a complex structure on its complement in $su(1,1)$, with $\mathfrak{t}_+(\mathfrak{t}_-)$ as (anti)-holomorphic directions.

The Killing form on $su(1,1)$ is

$$\kappa_{ij} = \kappa(\mathfrak{t}_i, \mathfrak{t}_j) = \frac{1}{2}\varepsilon_{ikm}\varepsilon_j{}^{km} = \eta_{ij}, \tag{4.2.19}$$

where η is the 'mostly minus' Minkowski metric:

$$\eta = \mathrm{diag}(1, -1, -1). \tag{4.2.20}$$

This Killing form is the same as the $\lambda = 1$ case of (2.2.5) as the sign difference between $\mathfrak{t}_0$ and t_0 cancels out. Following (2.2.13) we parametrise an $SU(1,1)$ matrix h using complex coordinates $(z_1, z_2) \in \mathbb{C}^{1,1}$ as

$$h = \begin{pmatrix} z_1 & \bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \qquad |z_1|^2 - |z_2|^2 = 1, \tag{4.2.21}$$

so that we can view $SU(1,1)$ as a submanifold of $\mathbb{C}^{1,1}$:

$$SU(1,1) = \{(z_1, z_2) \in \mathbb{C}^{1,1} | |z_1|^2 - |z_2|^2 = 1\}. \tag{4.2.22}$$

It is the double cover of AdS_3, the real submanifold of $\mathbb{C}^{1,1}$ defined[1] by

$$\mathrm{AdS}_3 = \{(Z_1, Z_2) \in \mathbb{C}^{1,1} | |Z_1|^2 - |Z_2|^2 = \ell^2\}/\mathbb{Z}_2, \tag{4.2.23}$$

with ℓ called the AdS length. The $\mathbb{Z}_2$ quotient identifies $(-Z_1, -Z_2)$ and (Z_1, Z_2).

The left invariant one-forms on $SU(1,1)$ defined via

$$h^{-1}dh = \varsigma^i \mathsf{t}_i = \varsigma^0 \mathsf{t}_0 + \varsigma^1 \mathsf{t}_1 + \varsigma^2 \mathsf{t}_2, \tag{4.2.24}$$

satisfy

$$d\varsigma^i + \frac{1}{2}\varepsilon^i{}_{jk}\varsigma^j \wedge \varsigma^k = 0. \tag{4.2.25}$$

We again make use of the complex combinations

$$\varsigma = \varsigma^1 + i\varsigma^2, \quad \bar{\varsigma} = \varsigma^1 - i\varsigma^2, \tag{4.2.26}$$

for which we have that

$$d\varsigma = -i\varsigma \wedge \varsigma^0, \quad d\varsigma^0 = \frac{i}{2}\bar{\varsigma} \wedge \varsigma. \tag{4.2.27}$$

In terms of the complex coordinates we find that

$$\varsigma = \varsigma^1 + i\varsigma^2 = 2i(z_1 dz_2 - z_2 dz_1), \quad \varsigma^0 = 2i(\bar{z}_2 dz_2 - \bar{z}_1 dz_1). \tag{4.2.28}$$

The dual left-invariant vector fields $\mathfrak{X}_i$, $i = 0, 1, 2$, generate the right action $h \to h\mathsf{t}_i$ and satisfy

$$[\mathfrak{X}_i, \mathfrak{X}_j] = \varepsilon_{ij}{}^k \mathfrak{X}_k, \tag{4.2.29}$$

so that the complex linear combinations, $\mathfrak{X}_\pm = \mathfrak{X}_1 \pm i\mathfrak{X}_2$, satisfy

$$[\mathfrak{X}_+, \mathfrak{X}_-] = -2i\mathfrak{X}_0, \quad [\mathfrak{X}_0, \mathfrak{X}_\pm] = \pm i\mathfrak{X}_\pm. \tag{4.2.30}$$

[1] Here we have used Z_i for these complex coordinates to distinguish the z's which are normalised from the AdS_3 coordinates which are not.

In terms of the complex coordinates they are

$$\mathfrak{X}_0 = \frac{i}{2}(z_2\partial_2+z_1\partial_1-\bar{z}_2\bar{\partial}_2-\bar{z}_1\bar{\partial}_1), \quad \mathfrak{X}_- = \bar{\mathfrak{X}}_+ = \mathfrak{X}_1-i\mathfrak{X}_2 = -i(\bar{z}_1\partial_2+\bar{z}_2\partial_1). \quad (4.2.31)$$

The only non-zero pairings are

$$\varsigma^0(\mathfrak{X}_0) = 1, \quad \varsigma(\mathfrak{X}_-) = \bar{\varsigma}(\mathfrak{X}_+) = 2. \quad (4.2.32)$$

The Poincaré disk model of hyperbolic two-space can also be viewed as the coset space

$$H^2 \simeq SU(1,1)/U(1), \quad (4.2.33)$$

where we consider the $U(1)$ generated by $\mathfrak{t}_0$. As H^2 is contractible this means that $SU(1,1)$ is a trivial circle bundle over H^2, with $\mathfrak{X}_0$ generating translation in the fibre direction.

We can construct a projection from the group manifold $SU(1,1)$ to H^2, using the complex coordinate $z \in \mathbb{C}$ on H^2, and in terms of the complex coordinates (z_1, z_2) for $SU(1,1)$ this projection is

$$\rho : SU(1,1) \to H^2, \quad h \mapsto z = \frac{z_2}{z_1}. \quad (4.2.34)$$

A global section of this bundle is given by

$$s : H^2 \to SU(1,1), \quad z \mapsto \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix}. \quad (4.2.35)$$

4.2.3 Hyperbolic vortices as Cartan connections

We know from Theorem 2.3 that the hyperbolic vortex equations can be interpreted as the flatness conditions for a $su(1,1)$ Cartan connection which encodes the geometry, modified by the vortices. In terms of the conventions of this Chapter the $\lambda = \lambda_0 = 1$ case of Theorem 2.3 becomes:

Proposition 4.1

The frame (4.2.3) *and the spin connection* (4.2.5) *for* H^2 *can be combined into the*

$su(1,1)$ valued gauge potential

$$\hat{A} = \Gamma \mathfrak{t}_0 + \frac{1}{2i} \left(e\mathfrak{t}_- - \bar{e}\mathfrak{t}_+ \right). \tag{4.2.36}$$

The flatness condition for $\hat{A}$ is equivalent to the structure equation (2.3.2) and Gauss equation (3.3.5) on H^2. The flatness of the pull-back potential $f^*\hat{A}$, for a holomorphic map $f : H^2 \to H^2$, is equivalent to the hyperbolic vortex equations being satisfied by the pair (ϕ, a) defined through (4.2.9).

In other words, the gauge potential $\hat{A}$ defines a Cartan connection describing the hyperbolic geometry of the Poincaré disk model of two-dimensional hyperbolic space while $f^*\hat{A}$ is the gauge potential for a Cartan connection describing the deformed geometry defined by the hyperbolic vortex (ϕ, a).

4.3 Vortices on $SU(1,1)$

4.3.1 Vortex equations and flat $SU(1,1)$ connections

We now define and solve vortex equations on the group manifold of $SU(1,1)$ and show that three-dimensional vortex configurations which solve them are equivariant versions of hyperbolic vortices. These are the Lorentzian analogues of the vortex configurations on $SU(2)$ from Definition 3.3.

To prepare for the equivariant description, we define the space of equivariant functions on $SU(1,1)$ as

$$C^\infty(SU(1,1), \mathbb{C})_N = \{F : SU(1,1) \to \mathbb{C} | 2i\mathfrak{X}_0 F = -NF\}, \quad N \in \mathbb{Z}, \tag{4.3.1}$$

with N called the equivariant degree of the function. One finds that

$$i((1-|z|^2)\bar{\partial} - \frac{N}{2}z)(s^*F)(z) = s^*(\mathfrak{X}_+ F), \tag{4.3.2}$$

where $F \in C^\infty(SU(1,1), \mathbb{C})_N$. This result is Lemma A.1 which we prove in Appendix A. From this we deduce the following commutative diagram

$$\begin{array}{ccc} C^\infty(SU(1,1),\mathbb{C})_N & \xrightarrow{\mathfrak{X}_+} & C^\infty(SU(1,1),\mathbb{C})_{N+2} \\ {\scriptstyle s^*}\Big\downarrow & & \Big\downarrow{\scriptstyle s^*} \\ C^\infty(H^2) & \xrightarrow[i((1-|z|^2)\bar\partial-\frac{N}{2}z)]{} & C^\infty(H^2). \end{array} \tag{4.3.3}$$

We will see later that functions with a well defined equivariant degree on $SU(1,1)$ can be used to construct the lift of a vortex of finite charge from H^2; it is these lifts of finite charge vortices that are the analogues of the vortex configurations on $SU(2)$.

Definition 4.2

A pair (Φ, A) of a one-form A on $SU(1,1)$ and a map $\Phi : SU(1,1) \to \mathbb{C}$ is called a vortex configuration on $SU(1,1)$ if it satisfies the vortex equations

$$(d\Phi - iA\Phi) \wedge \varsigma = 0, \qquad F_A = -\frac{i}{2}\left(|\Phi|^2 - 1\right) \varsigma \wedge \bar\varsigma, \tag{4.3.4}$$

where $F_A = dA$.

Note, that unlike in the Euclidean case considered in [10], we have not imposed any normalisation condition on A and have not fixed the equivariant degree of Φ. However, as observed in Chapter 3 the vortex equations imply the following equivariance condition:

$$\mathcal{L}_{\mathfrak{X}_0} A = d(A(\mathfrak{X}_0)), \quad i\mathcal{L}_{\mathfrak{X}_0}\Phi = -A(\mathfrak{X}_0)\Phi. \tag{4.3.5}$$

The first follows from the Cartan formula $\mathcal{L}_{\mathfrak{X}_0} = d\iota_{\mathfrak{X}_0} + \iota_{\mathfrak{X}_0} d$ and the form of $F_A = dA$ dictated by the second vortex equation. The equivariance condition for Φ can be obtained by contracting the first vortex equation with $(\mathfrak{X}_0, \mathfrak{X}_-)$. We discuss the case of Φ having equivariant degree $2N - 2$, the analogue of the $SU(2)$ case, later.

The vortex equations (4.3.4) clearly resemble the hyperbolic vortex equations, (4.2.8), with the complexified left-invariant one-forms, ς and $\bar\varsigma$ replacing the complexified frame, e and $\bar{e}$. We will establish the precise relation between the two at the end of this section.

We now come to the central result of this section, which is a three-dimensional analogue of the description of hyperbolic vortices in terms of a flat $SU(1,1)$ connection given in Proposition 4.1. It is also the Lorentzian analogue of Theorem 3.4 in Chapter 3, but differs from it in two important respects. In the Euclidean version, the relevant $U(1)$ bundle is the Hopf bundle, and associated line bundles are classified by an integer degree, but the total space $SU(2)$ is simply-connected. Here, the $U(1)$ bundle is trivial, but the total space $SU(1,1)$ is not simply connected. The generator of the fundamental group is the curve

$$\gamma = \{e^{\varphi \mathfrak{t}_0} \in SU(1,1) | \varphi \in [0, 4\pi)\}, \tag{4.3.6}$$

which enters our condition for vortex configuration to be globally solvable.

Theorem 4.3

A vortex configuration on $SU(1,1)$ determines a gauge potential for a flat $SU(1,1)$ connection on $SU(1,1)$ through the following expression:

$$\mathcal{A} = (A + \varsigma^0)\mathfrak{t}_0 + \frac{1}{2}\left(\Phi\varsigma\mathfrak{t}_- + \bar{\Phi}\bar{\varsigma}\mathfrak{t}_+\right). \tag{4.3.7}$$

Conversely, any flat $SU(1,1)$ connection $\mathcal{A}$ on $SU(1,1)$ with

$$\mathcal{A}(\mathfrak{X}_0) = p\mathfrak{t}_0, \quad \mathcal{A}(\mathfrak{X}_-) = \alpha\mathfrak{t}_0 + \Phi\mathfrak{t}_-, \tag{4.3.8}$$

for functions $p : SU(1,1) \to \mathbb{R}$ and $\alpha, \Phi : SU(1,1) \to \mathbb{C}$, determines a vortex configuration (Φ, A) via the expansion (4.3.7).

A gauge potential $\mathcal{A}$ for a flat $su(1,1)$ connection on $SU(1,1)$ of the form (4.3.7) which satisfies

$$\int_\gamma \mathcal{A} = 4\pi n\mathfrak{t}_0, \tag{4.3.9}$$

for some $n \in \mathbb{Z}$, can be trivialised as $\mathcal{A} = V^{-1}dV$, where $V : SU(1,1) \to SU(1,1)$ is a bundle map covering a holomorphic map $f : H^2 \to H^2$. Without loss of generality *we can write V in the form*

$$V : (z_1, z_2) \mapsto \frac{1}{\sqrt{|F_1|^2 - |F_2|^2}} \begin{pmatrix} F_1 & \bar{F}_2 \\ F_2 & \bar{F}_1 \end{pmatrix}, \tag{4.3.10}$$

where F_1 and F_2 are maps $SU(1,1) \to \mathbb{C}$, with $|F_1| > |F_2|$ and in particular $F_1 \neq 0$. The vortex configuration (Φ, A) can be computed from the bundle map V through

$$V^*\varsigma = \Phi\varsigma, \qquad A = V^*\varsigma^0 - \varsigma^0. \tag{4.3.11}$$

The proof is fairly similar to that of Theorem 3.4. The main difference being that the flat connection requires a prescribed holonomy around γ, (4.3.9), to be globally trivialised.

Proof. The proof that the vortex equations (4.3.4) imply flatness of $\mathcal{A}$ given in (4.3.7) is a simple calculation, see the proof of Theorem 3.4 in Chapter 3. Conversely, expanding an $su(1,1)$-valued one-form $\mathcal{A}$ on $SU(1,1)$ in terms of the generators $\mathfrak{t}_0, \mathfrak{t}_+$ and $\mathfrak{t}_-$, with coefficients which are linear combinations of the one-forms ς^0, ς and $\bar{\varsigma}$, and imposing (4.3.8) leads to a gauge potential $\mathcal{A}$ of the form (4.3.7) with Higgs field Φ and abelian gauge field

$$A = (p-1)\varsigma^0 + \frac{1}{2}\left(\alpha\varsigma + \bar{\alpha}\bar{\varsigma}\right). \tag{4.3.12}$$

The flatness of $\mathcal{A}$ then give the vortex equations (4.3.4), as already noted.

A connection $\mathcal{A}$ on $SU(1,1)$ can be trivialised in terms of $V : SU(1,1) \to SU(1,1)$ as $\mathcal{A} = V^{-1}dV$ if its path-ordered exponential is path-independent. In that case one can construct V explicitly from the path-ordered exponential of $\mathcal{A}$ along any path, starting at a fixed (but arbitrary) base point, see for example [40].

In our case, the flatness of $\mathcal{A}$ ensures the path-independence of the path-ordered exponential for all contractible paths on $SU(1,1)$, by the non-abelian Stokes Theorem. The condition (4.3.8) implies that the path-ordered exponential and the exponential of the ordinary integral of $\mathcal{A}$ along γ coincide, and finally (4.3.9) ensures that the path-ordered exponential of $\mathcal{A}$ along γ is trivial:

$$\mathcal{P}\exp(\int_\gamma \mathcal{A}) = \mathbb{I}. \tag{4.3.13}$$

Again using flatness of $\mathcal{A}$ we conclude that the path-ordered exponential along any closed curve on $SU(1,1)$ is the identity, thus establishing the path-independence of the path-ordered exponential.

It remains to show that the requirements (4.3.8) for $\mathcal{A} = V^{-1}dV$ force V to be a bundle map covering a holomorphic map $H^2 \to H^2$. The first condition becomes

$$\mathfrak{X}_0 V = pV\mathfrak{t}_0, \tag{4.3.14}$$

for $p : SU(1,1) \to \mathbb{R}$. However, this is precisely the infinitesimal formulation of the requirement that V preserves the fibres of the fibration $SU(1,1) \to H^2$, i.e., that V is a bundle map.

The second condition in (4.3.8) complex conjugates to

$$V^{-1}\mathfrak{X}_+ V = \bar{\alpha}\mathfrak{t}_0 + \bar{\Phi}\mathfrak{t}_+. \tag{4.3.15}$$

Applying

$$V^{-1}dV = V^*\varsigma^0\mathfrak{t}_0 + \frac{1}{2}\left(V^*\varsigma\mathfrak{t}_- + V^*\bar{\varsigma}\mathfrak{t}_+\right) \tag{4.3.16}$$

to $\mathfrak{X}_+$, the condition (4.3.15) is thus equivalent to the vanishing of the t_--component in $V^{-1}dV$:

$$V^*\varsigma(\mathfrak{X}_+) = 0. \tag{4.3.17}$$

We need to show that this is equivalent to V covering a holomorphic map.

Using the parameterisation (4.3.10) of V, we see from (4.3.14) that the components of V satisfy

$$\mathfrak{X}_0\left(\frac{F_i}{\sqrt{|F_1|^2 - |F_2|^2}}\right) = \frac{i}{2}p\left(\frac{F_i}{\sqrt{|F_1|^2 - |F_2|^2}}\right). \tag{4.3.18}$$

It follows that the map $F = \pi \circ V = F_2/F_1$ has equivariant degree zero, and that V

covers the map

$$f = s^*\left(\frac{F_2}{F_1}\right) : H^2 \to H^2. \tag{4.3.19}$$

Applying (4.3.2) to the map $F = F_2/F_1$ we deduce that f being holomorphic is equivalent to

$$\mathfrak{X}_+\left(\frac{F_2}{F_1}\right) = 0. \tag{4.3.20}$$

However, again recalling that $F_1 \neq 0$ and using the explicit form (4.2.28) of ς, this is seen to be equivalent to the condition (4.3.14) on V. Thus, the requirement (4.3.8) forces V to be a bundle map covering a holomorphic map, as claimed. □

The Theorem and its proof deserve some comments. First we note that the vortex equations (4.3.4) are invariant under $U(1)$ gauge transformations of the form

$$(\Phi, A) \mapsto (e^{i\beta}\Phi, A + d\beta), \quad \beta \in C^\infty(SU(1,1)). \tag{4.3.21}$$

The $U(1)$ gauge invariance is implemented at the level of the bundle map V via

$$V \mapsto \tilde{V} = Ve^{\beta \mathfrak{t}_0}, \quad \beta \in C^\infty(SU(1,1)). \tag{4.3.22}$$

This new trivialisation defines the same f as V and leads to a connection $\tilde{V}^{-1}d\tilde{V}$ differing from $\mathcal{A} = V^{-1}dV$ by the $U(1)$ gauge transformation given in (4.3.21).

Secondly, we observe that, since $SU(1,1)$ is a trivial $U(1)$ bundle, we can build vortex configurations on $SU(1,1)$ from a given holomorphic map $f : H^2 \to H^2$ by choosing

$$F_1(z_1, z_2) = 1, \qquad F_2(z_1, z_2) = f\left(\frac{z_2}{z_1}\right). \tag{4.3.23}$$

For this choice of V the connection satisfies $\mathcal{A}(\mathfrak{X}_0) = 0$ and also, by flatness, $\mathcal{L}_{\mathfrak{X}_0}\mathcal{A} = 0$, so that $\mathcal{A}$ is constant along the fibre direction. This is in contrast to the case of the previous Chapter where the fact that the bundle is non-trivial precluded such a global lift.

Finally, the condition (4.3.9) is needed to ensure the existence of a globally defined trivialisation of $\mathcal{A}$. When it is violated, one can still trivialise, but one will in general have to work with local trivialisations, defined in at least two simply connected patches which cover $SU(1,1)$. We will not consider such trivialisations in this Chapter.

4.3.2 Vortex configurations of finite equivariant degree

In the following we exhibit a choice of bundle map which is a natural lift of the Blaschke product considered in Sect. 4.2.1. The resulting vortices on $SU(1,1)$ are lifts of hyperbolic vortices with a finite number of zeros, and a Lorentzian analogue of the vortex configurations obtained from homogeneous polynomials in the previous Chapter.

Before stating our result we define functions $F_1, F_2 : SU(1,1) \to \mathbb{C}$ for given complex numbers c_k, $k = 1, \ldots, N$, in the unit disk via

$$F_1 = \prod_{k=1}^{N}(z_1 - \bar{c}_k z_2), \qquad F_2 = \prod_{k=1}^{N}(z_2 - c_k z_1), \tag{4.3.24}$$

and note that F_2/F_1 is a function of $z = z_2/z_1$ and given by the Blaschke product (4.2.11). In particular, therefore $|F_1| > |F_2|$, and we can use F_1, F_2 to define a bundle map V covering the Blaschke product (4.2.11) via (4.3.10).

Corollary 4.4
Let $V : SU(1,1) \to SU(1,1)$ be the bundle map (4.3.10) constructed from (4.3.24). Then the vortex configuration (Φ, A) constructed from the flat connection $\mathcal{A} = V^{-1}dV$ via Theorem 4.3 has a Higgs field of equivariant degree $2N-2$ and a gauge field which satisfies the normalisation condition

$$A(\mathfrak{X}_0) = N - 1. \tag{4.3.25}$$

The vortex configuration (Φ, A) can be given in terms of F_1, F_2 as

$$\Phi = \frac{F_1 \partial_2 F_2 - F_2 \partial_2 F_1}{z_1(|F_1|^2 - |F_2|^2)}, \tag{4.3.26}$$

and

$$A = (N-1)\varsigma^0 - \frac{i}{2}\mathfrak{X}_- \ln D^2 \varsigma + \frac{i}{2}\mathfrak{X}_+ \ln D^2 \bar{\varsigma}, \tag{4.3.27}$$

where $D^2 = |F_1|^2 - |F_2|^2$.

Proof. It follows from the explicit form of F_1, F_2 that

$$\mathfrak{X}_0 V = NVt_0, \tag{4.3.28}$$

so that $\mathcal{A}(\mathfrak{X}_0) = N\mathfrak{t}_0$ and therefore, by the decomposition (4.3.7), $A(\mathfrak{X}_0) = N - 1$. The general equivariance condition (4.3.5) then implies

$$i\mathcal{L}_{\mathfrak{X}_0}\Phi = -(N-1)\Phi, \tag{4.3.29}$$

so that Φ has equivariant degree $2N-2$ by (4.3.1). To get the explicit expression for Φ we compute

$$\Phi = \frac{1}{2}V^*\varsigma(\mathfrak{X}_-). \tag{4.3.30}$$

To compute A we use

$$V^*\varsigma^0(\mathfrak{X}_0) = N, \quad V^*\varsigma^0(\mathfrak{X}_+) = i\mathfrak{X}_+ \ln D^2, \quad V^*\varsigma^0(\mathfrak{X}_-) = -i\mathfrak{X}_- \ln D^2, \tag{4.3.31}$$

and (4.3.12) to get the claimed result. □

Recall that the Blaschke product (4.2.11) gives rise, via the pull-back construction, to a hyperbolic vortex of charge $N-1$ on H^2. The Corollary above shows that the natural covering V of the Blaschke product defines a vortex configuration on $SU(1,1)$ of finite equivariant degree $2N-2$.

For finite Blaschke products, the lift (4.3.24) defines a natural bundle map which we used to construct a three-dimensional vortex configuration with non-zero degree. In general, lifting to a configuration with non-zero equivariant degree is not obvious. For example, the vortices on the hyperbolic cylinder studied in [16, 28] require the infinite Blaschke product

$$f(z) = z\prod_{j=1}^{\infty}\left(\frac{z-a_j^2}{1-a_j^2 z}\right)^2, \tag{4.3.32}$$

where the zeros of f are at $a_j = i\tanh\left(\frac{j\lambda}{2}\right)$ with λ defined in [16, 28] as $\lambda = \frac{\pi K'(k)}{K(k)}$ for K the elliptic integral of the first kind, $K'(k) = K(\sqrt{1-k^2})$ and any $0 < k < 1$. These can still be lifted to $SU(1,1)$ via the lift (4.3.23), but there does not appear to be any, natural, non-trivial option.

4.3.3 Lifting Cartan connections for hyperbolic vortices

We have already seen how to lift hyperbolic vortices to vortex configurations on $SU(1,1)$. Since the latter can be expressed in terms of a flat $SU(1,1)$ connection, it is natural to expect a link with the Cartan connection encoding hyperbolic vortices according to Proposition 4.1. In this short section, we exhibit this link.

For a nowhere-vanishing function $g : H^2 \to \mathbb{C}$ define the map

$$r_g : H^2 \to SU(1,1), \quad r_g = \begin{pmatrix} \frac{\bar{g}}{|g|} & 0 \\ 0 & \frac{g}{|g|} \end{pmatrix}. \tag{4.3.33}$$

We use this map as a gauge transform in the following Lemma, which is a Lorentzian analogue of Lemma 3.11 in Chapter 3.

Lemma 4.5

With the section $s : H^2 \to SU(1,1)$ defined as in (4.2.35)*, the gauge potential* (4.2.36) *for the Cartan connection of the hyperbolic disk is trivialised by s:*

$$\hat{A} = s^{-1}ds. \tag{4.3.34}$$

If V is a bundle map of the form (4.3.10) *covering a holomorphic map $f : H^2 \to H^2$, then the gauge potential $f^*\hat{A}$ for the deformed Cartan geometry and the pull-back via s of $\mathcal{A} = V^{-1}dV$ are related through the gauge transformation r_{f_1}, where $f_1 = F_1 \circ s$:*

$$f^*\hat{A} = r_{f_1}^{-1} s^*(\mathcal{A}) r_{f_1} + r_{f_1}^{-1} dr_{f_1}. \tag{4.3.35}$$

Note that, if we work in the gauge where $F_1 = 1$ and $F_2 = f\left(\frac{z_2}{z_1}\right)$, then $r_{f_1} = \mathbb{I}$, and $f^*\hat{A}$ agrees with $s^*\left(V^{-1}dV\right)$. More generally, the fact that F_1, and hence f_1, has no zeros means that the gauge transformation r_{f_1} is smooth. This is in contrast to the Euclidean case of the previous Chapter where a singular gauge transformation was needed.

Proof. The proof is a straightforward calculation which proceeds along the same lines as Lemma 3.11. To show (4.3.34) one uses (4.2.35) and compares to the definitions of e and Γ in terms of the complex coordinates in (4.2.3) and (4.2.5). To show (4.3.35), one notes $s \circ f = (V \circ s) r_{f_1}$ and $\mathcal{A} = V^{-1} dV$. $\square$

4.4 Magnetic Dirac operators on $\widetilde{\mathrm{AdS}}_3$ and Minkowski space

4.4.1 Notational conventions

We denote three-dimensional Minkowski space by $\mathbb{R}^{1,2}$, and use a 'mostly minus' Lorentzian metric η with matrix (4.2.20) in an orthonormal basis. We write elements of Minkowski space as $\vec{x} = (x^0, x^1, x^2)^t$ so that

$$\eta = (dx^0)^2 - (dx^1)^2 - (dx^2)^2. \tag{4.4.1}$$

Our volume element is $dx^0 \wedge dx^1 \wedge dx^2$ so that dx^0, dx^1, dx^2 is an oriented basis of the cotangent space. We use indices $i, j \ldots$ in the range $0, 1, 2$, raised and lowered using η_{ij}. The scalar and vector product are given by

$$\vec{x} \cdot \vec{y} = x^i y_i, \qquad (\vec{x} \times \vec{y})^k = \varepsilon_{ij}{}^k x^i y^j, \tag{4.4.2}$$

for $\vec{x}, \vec{y} \in \mathbb{R}^{1,2}$, $\varepsilon_{012} = 1$ and the summation convention being understood between pairs of raised and lowered indices. The Lorentzian length squared of $\vec{x}$ is denoted by

$$r^2 = x^i x_i = (x^0)^2 - (x^1)^2 - (x^2)^2. \tag{4.4.3}$$

We also use the notation $\partial_i = \partial / \partial x^i$ for partial derivatives.

On $SU(1,1)$ we continue to work with the notation introduced in Sect. 4.2.2, and use the following oriented orthonormal frame consisting of

$$\frac{1}{2}\varsigma^0, \frac{1}{2}\varsigma^1, \frac{1}{2}\varsigma^2, \tag{4.4.4}$$

the metric

$$ds^2 = \frac{1}{4}\left((\varsigma^0)^2 - (\varsigma^1)^2 - (\varsigma^2)^2\right), \tag{4.4.5}$$

and orientation

$$\mathrm{Vol}_{\mathrm{AdS}} = \frac{1}{8}\varsigma^0 \wedge \varsigma^1 \wedge \varsigma^2. \tag{4.4.6}$$

Differential forms provide the natural language for our discussion, but occasionally we use the isomorphisms between forms and vector fields which are possible on a three-dimensional manifold with a non-degenerate inner product and volume form. The inner product allows one to identify vector fields with one-forms; denoting the volume form Vol, it establishes a bijection between a vector field X and a two-form F via

$$\iota_X \mathrm{Vol} = F. \tag{4.4.7}$$

On $SU(1,1)$, for example, the vector field $\mathfrak{X}_0$ generating the fibre translation is mapped to the two-form $\frac{1}{8}\varsigma^1 \wedge \varsigma^2$ via (4.4.6), and this will play a role in our discussions as the analogue of the background field of the previous Chapter.

In the following we will be considering the Dirac equation on both $SU(1,1)$ and $\mathbb{R}^{1,2}$, to this end we need to fix our conventions for the Clifford algebra $Cl(1,2)$. The algebra is generated by the gamma matrices, γ_i which satisfy

$$\{\gamma_i, \gamma_j\} = -2\eta_{ij}. \tag{4.4.8}$$

We pick $\gamma_i = 2\mathfrak{t}_i$ as our representation of the gamma matrices.

4.4.2 Stereographic projection and frames

We will be using a stereographic projection to relate Dirac operators on $\widetilde{\mathrm{AdS}}_3 \simeq SU(1,1)$ to Dirac operators on Minkowski space. We now set up our conventions and explain how orthonormal frames on these spaces are mapped into each other via stereographic projection.

To discuss stereographic projection from $\widetilde{\mathrm{AdS}}_3$ to $\mathbb{R}^{1,2}$, it is helpful to think of $\widetilde{\mathrm{AdS}}_3$ as a real manifold. As a subspace of $\mathbb{R}^{2,2}$, and with AdS length ℓ, it is given by

$$\widetilde{\mathrm{AdS}}_3 = \{(y^0, y^1, y^2, y^3) \in \mathbb{R}^{2,2} | (y^0)^2 - (y^1)^2 - (y^2)^2 + (y^3)^2 = \ell^2\}. \tag{4.4.9}$$

This is just the definition of $\widetilde{\mathrm{AdS}}_3$ as a submanifold of $\mathbb{C}^{1,1}$ from (4.2.23) written in terms of real coordinates. The real coordinates are related to the complex coordinates for $SU(1,1)$, (4.2.21), through

$$\ell(z_1, z_2) = (y^3 + iy^0, y^2 - iy^1). \tag{4.4.10}$$

Just as stereographic projection from the sphere requires one to single out a north and a south pole, we need to pick special points on $\widetilde{\mathrm{AdS}}_3$ to define our stereographic projection. We choose

$$P_\pm = (0,0,0,\pm\ell) \in \widetilde{\mathrm{AdS}}_3. \tag{4.4.11}$$

Then, to map a point $(y^0, y^1, y^2, y^3) \in \widetilde{\mathrm{AdS}}_3$ into $\mathbb{R}^{1,2}$, we draw the line between it and the point P_-; the image is the intersection with $\mathbb{R}^{1,2}$ at $\{y^3 = 0\}$, see Fig. 4.2. This intersection does not exist for all points in $\widetilde{\mathrm{AdS}}_3$ but only for the subset

$$\widetilde{\mathrm{AdS}}_+ = \{(y^0, y^1, y^2, y^3) \in \widetilde{\mathrm{AdS}}_3 | y^3 > -\ell\}. \tag{4.4.12}$$

Moreover, the point of intersection necessarily lies in the subset $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ defined as

$$\mathcal{I}_\ell = \{(x^0, x^1, x^2) \in \mathbb{R}^3 | r^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 > -\ell^2\}. \tag{4.4.13}$$

Geometrically $\mathcal{I}_\ell$ is the inside of a single sheeted hyperboloid.

Thus we can define the stereographic projection map

$$\mathrm{St} : \widetilde{\mathrm{AdS}}_+ \to \mathbb{R}^{1,2}, \quad (y^0, y^1, y^2, y^3) \mapsto \vec{x} = \left(\frac{\ell y^0}{\ell + y^3}, \frac{\ell y^1}{\ell + y^3}, \frac{\ell y^2}{\ell + y^3} \right) \tag{4.4.14}$$

and its inverse

$$\mathrm{St}^{-1} : \mathcal{I}_\ell \subset \mathbb{R}^{1,2} \to \widetilde{\mathrm{AdS}}_3, \quad \vec{x} \mapsto (y^0, y^1, y^2, y^3) = \frac{\ell}{\ell^2 + r^2} \left(2\ell x^0, 2\ell x^1, 2\ell x^2, \ell^2 - r^2 \right). \tag{4.4.15}$$

For calculations, it is helpful to express some of these maps in matrix notation. Using $\vec{\mathfrak{t}} = (\mathfrak{t}_0, \mathfrak{t}_1, \mathfrak{t}_2)^t$, we identify the point $(y^0, y^1, y^2, y^3) \in \widetilde{\mathrm{AdS}}_3$ with the $SU(1,1)$ matrix

$$M(y^0, y^1, y^2, y^3) = \frac{1}{\ell}(y^3 \mathbb{I} + 2\vec{y} \cdot \vec{\mathfrak{t}}). \tag{4.4.16}$$

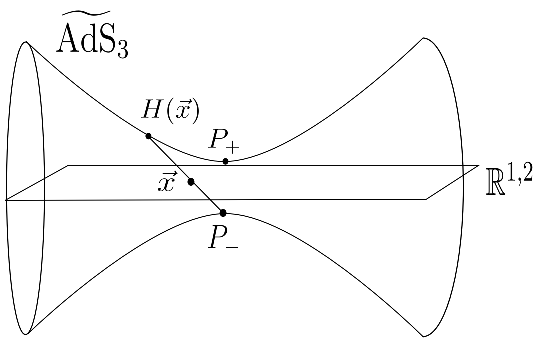

Figure 4.2: A schematic picture of the stereographic projection from $\widetilde{\mathrm{AdS}}_3$ to $\mathbb{R}^{1,2}$, with one dimension suppressed; we used the notation introduced in (4.4.11)

Up to scaling by ℓ, the inverse stereographic projection can then be written as

$$
\begin{aligned}
H :& \mathcal{I}_\ell \subset \mathbb{R}^{1,2} \to SU(1,1), \\
& \vec{x} \mapsto \frac{\ell^2 - r^2}{\ell^2 + r^2}\mathbb{I} + \frac{4\ell}{\ell^2 + r^2}\vec{x} \cdot \vec{\mathfrak{t}} = \frac{1}{\ell^2 + r^2}\begin{pmatrix} \ell^2 - r^2 + 2i\ell x^0 & 2i\ell(x^1 - ix^2) \\ -2i\ell(x^1 + ix^2) & \ell^2 - r^2 - 2i\ell x^0 \end{pmatrix}.
\end{aligned}
\tag{4.4.17}
$$

In stereographic coordinates, the bundle projection $\rho : SU(1,1) \to H^2$ therefore becomes

$$
\rho \circ H : \vec{x} \mapsto \frac{z_2}{z_1} = -i\frac{2\ell(x^1 + ix^2)}{\ell^2 - r^2 + 2i\ell x^0}. \tag{4.4.18}
$$

We note that, in terms of (4.4.16), the condition $y^3 > -\ell$ can be written as $\mathrm{tr}(M) > -2$, and that matrices which satisfy this condition lie in the image of the exponential map.

We will also need a Lorentzian version of the gnomonic projection discussed and used in the previous Chapter. This is the map $G : \mathcal{I}_\ell \subset \mathbb{R}^{1,2} \to SU(1,1)$ given by

$$
G : \vec{x} \mapsto \frac{1}{\sqrt{\ell^2 + r^2}}\left(\ell\mathbb{I} + 2\vec{x} \cdot \vec{\mathfrak{t}}\right), \tag{4.4.19}
$$

and satisfying $G(\vec{x})^2 = H(\vec{x})$. The geometric interpretation of this result is shown in Fig. 4.3 and explained in the caption. Note also the map H is an AdS version of the projection relating a sheet of the two-sheeted hyperboloid to the disk in the Poincaré model. The map G is the AdS analogue of the map to the Beltrami disk model.

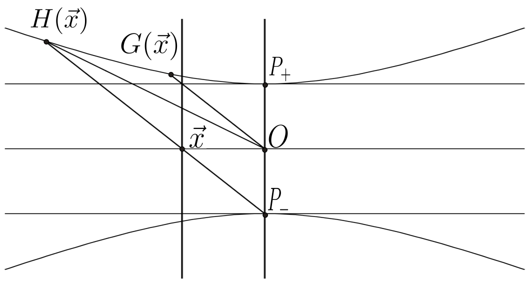

Figure 4.3: The lines $OG(\vec{x})$ and $P_-H(\vec{x})$ define the maps G and H. They are analogous to, respectively, the Beltrami and Poincaré map in hyperbolic geometry. The area bounded by the hyperbolic segment $P_+H(\vec{x})$ and the straight lines $OH(\vec{x})$ and OP_+ is twice that bounded by the hyperbolic segment $P_+G(\vec{x})$ and the lines $OG(\vec{x})$ and OP_+. This is the geometry underlying the relation $H(\vec{x}) = G^2(\vec{x})$.

By expanding $H^{-1}dH$ we define one-forms, ϑ_i $i = 0, 1, 2$, on $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ via

$$H^{-1}dH = \frac{1}{\Omega}\vec{\vartheta} \cdot \vec{\mathfrak{t}}, \quad \vartheta^i = \Omega H^* \varsigma^i, \tag{4.4.20}$$

where $\vec{\vartheta} = (\vartheta^0, \vartheta^1, \vartheta^2)^t$ and we used the scale factor

$$\Omega = \frac{\ell^2 + r^2}{4\ell}. \tag{4.4.21}$$

We find that

$$\vec{\vartheta} \cdot \vec{\mathfrak{t}} = \frac{1}{\ell^2 + r^2}\left(2(\vec{x} \cdot \vec{\mathfrak{t}})(\vec{x} \cdot d\vec{x}) + (\ell^2 - r^2)d\vec{x} \cdot \vec{\mathfrak{t}} - 2\ell(\vec{x} \times d\vec{x}) \cdot \vec{\mathfrak{t}}\right) = G^{-1}(d\vec{x} \cdot \vec{\mathfrak{t}})G. \tag{4.4.22}$$

This means that the ϑ^i, $i = 0, 1, 2$, give a Lorentz-rotated basis for the cotangent space of $\mathcal{I}_\ell$.

Lemma 4.6

The pullbacks of the Maurer-Cartan one-form via G and H are related through

$$H^{-1}dH = G^{-1}dG + G^{-1}(G^{-1}dG)G, \tag{4.4.23}$$

with the inverse relation expressed as

$$G^{-1}dG = \frac{1}{2}H^{-1}dH - \star\left(d\Omega \wedge H^{-1}dH\right). \tag{4.4.24}$$

Here $\star$ denotes the Hodge star on $SU(1,1)$ with respect to the orientation (4.4.6).

Proof. The proof follows by a calculation which is similar to the corresponding Euclidean version in the previous Chapter, but differs in important signs. The first statement (4.4.23) follows from the fact that $H = G^2$. For (4.4.24) we use (4.4.23) to write it as

$$\star\left(2d\Omega \wedge \left(dGG^{-1} + G^{-1}dG\right)\right) = -\left(dGG^{-1} - G^{-1}dG\right). \tag{4.4.25}$$

Then we compute

$$G^{-1}dG = \frac{2\left(\ell d\vec{x}\cdot\vec{\mathfrak{t}} - (\vec{x}\times d\vec{x})\cdot\vec{\mathfrak{t}}\right)}{\ell^2 + r^2}, \tag{4.4.26}$$

which gives us that

$$dGG^{-1} + G^{-1}dG = \frac{4\left(\ell d\vec{x}\cdot\vec{\mathfrak{t}}\right)}{\ell^2 + r^2}, \qquad dGG^{-1} - G^{-1}dG = \frac{4(\vec{x}\times d\vec{x})\cdot\vec{\mathfrak{t}}}{\ell^2 + r^2}. \tag{4.4.27}$$

With $2d\Omega = \frac{\vec{x}\cdot d\vec{x}}{\ell}$ and

$$\star\left(\vec{x}\cdot d\vec{x} \wedge d\vec{x}\cdot\vec{\mathfrak{t}}\right) = (d\vec{x}\times\vec{x})\cdot\vec{\mathfrak{t}} \tag{4.4.28}$$

we get (4.4.25) and hence (4.4.24). □

4.4.3 Magnetic Dirac operators

On $SU(1,1)$, a global gauge potential for the spin connection is given by

$$\Gamma_{SU(1,1)} = -\frac{1}{8}[\gamma_i, \gamma_j]\omega^{ij} = \frac{1}{2}h^{-1}dh. \tag{4.4.29}$$

The Dirac operator in the left-invariant frame is then

$$\not{D}_{SU(1,1)} = \frac{4}{\ell}\mathfrak{t}^i\mathfrak{X}_i - \frac{3}{2\ell}\mathbb{I} = \frac{2i}{\ell}\begin{pmatrix}\mathfrak{X}_0 & -\mathfrak{X}_- \\ \mathfrak{X}_+ & -\mathfrak{X}_0\end{pmatrix} - \frac{3}{2\ell}\mathbb{I}. \tag{4.4.30}$$

Minimal coupling to an abelian gauge potential $A = A_i \varsigma^i$ yields

$$\not{D}_{SU(1,1),A} = \frac{4}{\ell}\mathfrak{t}^i(\mathfrak{X}_i + iA_i) - \frac{3}{2\ell}\mathbb{I} = \frac{2i}{\ell}\begin{pmatrix} \mathfrak{X}_0 + iA_0 & -(\mathfrak{X}_- + iA_-) \\ \mathfrak{X}_+ + iA_+ & -(\mathfrak{X}_0 + iA_0) \end{pmatrix} - \frac{3}{2\ell}\mathbb{I}. \tag{4.4.31}$$

The Dirac operator on $\mathbb{R}^{1,2}$ minimally coupled to $\vec{A} \cdot d\vec{x}$ is

$$\not{D}_{\mathbb{R}^{1,2},A} = 2\mathfrak{t}^i(\partial_i + iA_i). \tag{4.4.32}$$

In the following, spinors Ψ which satisfy the massless Dirac equation $\not{D}_A \Psi = 0$ on either $SU(1,1)$ or $\mathbb{R}^{1,2}$ coupled to an abelian gauge potential are called magnetic Dirac modes, or simply magnetic modes. The following Lemma exhibits the relation between magnetic Dirac modes on $SU(1,1)$ and $\mathbb{R}^{1,2}$.

Lemma 4.7

If $\Psi : SU(1,1) \to \mathbb{C}^{1,1}$ is a magnetic mode of the Dirac operator on $SU(1,1)$ coupled to the $U(1)$ gauge field A then

$$\Psi_H = G\Omega^{-1}H^*\Psi \tag{4.4.33}$$

is a magnetic mode of the Dirac operator (4.4.32) on $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ coupled to the gauge potential H^*A.

Proof. The pull-back of the spin connection is

$$H^*\Gamma_{SU(1,1)} = \frac{1}{2}H^{-1}dH. \tag{4.4.34}$$

Using (4.4.23) one can show that

$$d + \frac{1}{2}H^{-1}dH = \Omega G^{-1}\left(d + \frac{1}{2}\left(GdG^{-1} + G^{-1}dG\right) + \Omega^{-1}d\Omega\right)\Omega^{-1}G. \tag{4.4.35}$$

Then combining (4.4.27) with

$$\Omega^{-1}d\Omega = \frac{2\vec{x}\cdot d\vec{x}}{\ell^2 + r^2}, \tag{4.4.36}$$

gives that

$$\mathfrak{t}^i \iota_{\partial_i}(\frac{1}{2}\left(GdG^{-1}+G^{-1}dG\right)+\Omega^{-1}d\Omega)=-\frac{2\vec{x}\cdot\vec{\mathfrak{t}}}{\ell^2+r^2}+\frac{2\vec{x}\cdot\vec{\mathfrak{t}}}{\ell^2+r^2}=0. \tag{4.4.37}$$

Using these results we compute the pull-back of the Dirac operator on $SU(1,1)$, coupled to both the spin connection and the abelian gauge potential A, to the flat frame in $\mathbb{R}^{1,2}$:

$$\begin{aligned}\frac{\ell}{2}H^*\not{D}_{SU(1,1),A} &= 2\mathfrak{t}^i\iota_{H^*\mathfrak{X}_i}\left(d+\frac{1}{2}H^{-1}dH+iH^*A\right),\\ &= \Omega G^{-1}2\mathfrak{t}^i\iota_{\partial_i}G\left(d+\frac{1}{2}H^{-1}dH+iH^*A\right),\\ &= \Omega^2G^{-1}2\mathfrak{t}^i\iota_{\partial_i}\left(d+iH^*A\right)\Omega^{-1}G.\end{aligned} \tag{4.4.38}$$

This implies the claimed relation between the magnetic modes of $\not{D}_{SU(1,1),A}$ and $\not{D}_{\mathbb{R}^{1,2},H^*A}$. □

4.5 Magnetic Dirac modes from vortices

4.5.1 Dirac modes on $SU(1,1)$

We now show how to obtain magnetic Dirac modes on $SU(1,1)$ from vortex configurations on $SU(1,1)$. To set the scene, we consider a simpler construction of magnetic modes from a holomorphic function $F : SU(1,1) \to \mathbb{C}$.

Proposition 4.8

Let $n \in \mathbb{N}$ and consider the gauge potential

$$A=-\frac{2n+1}{4}\varsigma^0 \tag{4.5.1}$$

and the homogeneous function $F_n=\sum_{k=0}^{n-1}a_k z_2^k z_1^{n-1-k}$ on $SU(1,1)$. Then the spinor

$$\Psi=\begin{pmatrix}F_n\\0\end{pmatrix} \tag{4.5.2}$$

solves the Dirac equation on $SU(1,1)$ minimally coupled to A.

Proof. First observe that

$$2i\mathfrak{X}_0 F_n = (1-n)F_n, \tag{4.5.3}$$

showing that F_n has equivariant degree $n-1$. Next note that $\mathfrak{X}_+ F_n = 0$, since F_n is holomorphic, that $A(\mathfrak{X}_+) = 0$ and that

$$(\mathfrak{X}_0 + iA(\mathfrak{X}_0))F_n = -\frac{3i}{4}F_n. \tag{4.5.4}$$

Using these and Equation (4.4.31), the equation $\not{D}_{SU(1,1),A}\Psi = 0$ reduces to

$$\frac{2i}{\ell}(\mathfrak{X}_0 + iA_0)F_n - \frac{3}{2\ell}F_n = 0 \tag{4.5.5}$$

so Ψ is indeed a magnetic mode. □

The following Definition and Theorem are similar to the Euclidean version from Chapter 3 . However, the non-linear equation in the Definition of a vortex magnetic mode has an important overall sign difference. We will encounter a generalisation of this Definition in the next Chapter, where the coefficient of $\Psi^\dagger h^{-1} dh \Psi$ can be $-1, 0$ or 1. This corresponds to vortex magnetic modes constructed from the Bradlow and Ambjorn-Olsen vortex equations on H^2.

Definition 4.9
A pair (Ψ, A) *of a spinor* Ψ *and a one-form* A *on* $SU(1,1)$ *is said to be a vortex magnetic mode of the Dirac equation on* $SU(1,1)$ *if*

$$\not{D}_{SU(1,1),A}\Psi = 0, \quad F_A = -\frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh \Psi - \frac{1}{4}\varsigma^1 \wedge \varsigma^2, \tag{4.5.6}$$

with $\star$ *the Hodge star operator on* $SU(1,1)$ *with respect to the metric* (4.4.5) *and orientation* (4.4.6).

We now give a result that enables the construction of magnetic modes from any vortex configuration.

Theorem 4.10

Let (Φ, A) be a vortex configuration on $SU(1,1)$. Then the pair (Ψ, A'), where

$$\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \qquad A' = -A - \frac{3}{4}\varsigma^0, \tag{4.5.7}$$

is a vortex magnetic mode on $SU(1,1)$.

Proof. The spinor is a magnetic mode of $\not{D}_{SU(1,1),A'}$ if

$$\left(i\mathfrak{X}_0 - A'_0 - \frac{3}{4}\right)\Phi = 0 \quad \text{and} \quad \mathfrak{X}_+\Phi + iA'_+\Phi = 0. \tag{4.5.8}$$

Now $A'_0 = A'(\mathfrak{X}_0) = -A(\mathfrak{X}_0) - \frac{3}{4}$ so the first equation follows from the equivariance condition (4.3.5). The second follows from (4.3.4) since $A'(\mathfrak{X}_+) = -A(\mathfrak{X}_+)$ and contracting (4.3.4) with $(\mathfrak{X}_+, \mathfrak{X}_-)$ leads to

$$\mathfrak{X}_+\Phi - iA(\mathfrak{X}_+)\Phi = 0. \tag{4.5.9}$$

For the non-linear equation with a spinor of the form given in (4.5.7) we have that

$$\frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh \Psi = \frac{4i}{\ell} \star |\Phi|^2 \left(\frac{i}{2}\varsigma^0\right) = -|\Phi|^2 \varsigma^1 \wedge \varsigma^2. \tag{4.5.10}$$

On the other hand using (4.3.4) gives

$$F_{A'} = -F_A + \frac{3}{4}\varsigma^1 \wedge \varsigma^2 = \left(|\Phi|^2 - \frac{1}{4}\right)\varsigma^1 \wedge \varsigma^2, \tag{4.5.11}$$

from which the non-linear equation follows. □

Both the magnetic two-forms $F_{A'}$ and F_A are proportional to $\varsigma^1 \wedge \varsigma^2$, with the factor of proportionality a function on $SU(1,1)$. The magnetic vector fields associated to both of them, via (4.4.7), are therefore similarly proportional to the vector field $\mathfrak{X}_0$, and so the magnetic field lines are just the fibres of the fibration $\rho : SU(1,1) \to H^2$.

To visualise these fibres in the embedding of $\widetilde{\mathrm{AdS}}_3$ in $\mathbb{R}^{2,2}$ (with one dimension suppressed), we note that they are in particular geodesics on $\widetilde{\mathrm{AdS}}_3$ and therefore can be obtained by intersections of the embedding (4.4.9) with planes in $\mathbb{R}^{2,2}$. The lower dimensional version of this, intersection of planes in $\mathbb{R}^3$ with AdS_2, was used to produce the picture of geodesics on AdS_2, embedded in $\mathbb{R}^{2,1}$, in Fig. 4.4.

We can write down geodesics on $\widetilde{\mathrm{AdS}}_3$ explicitly by expressing the right action of $e^{\alpha t_0}$, $\alpha \in [0, 4\pi)$, which generates them, in real coordinates. Using the parametrisation (4.4.16) the orbit of a point $(y^0, y^1, y^2, y^3) \in \widetilde{\mathrm{AdS}}_3$ is

$$\left(\left(y^0 \cos\frac{\alpha}{2} + y^3 \sin\frac{\alpha}{2}\right), \left(y^1 \cos\frac{\alpha}{2} - y^2 \sin\frac{\alpha}{2}\right), \left(y^2 \cos\frac{\alpha}{2} + y^1 \sin\frac{\alpha}{2}\right), \left(y^3 \cos\frac{\alpha}{2} - y^0 \sin\frac{\alpha}{2}\right)\right). \tag{4.5.12}$$

In other words moving along the fibre is equivalent to a rotation by $\frac{\alpha}{2}$ in both the y^3, y^0 and y^1, y^2 plane.

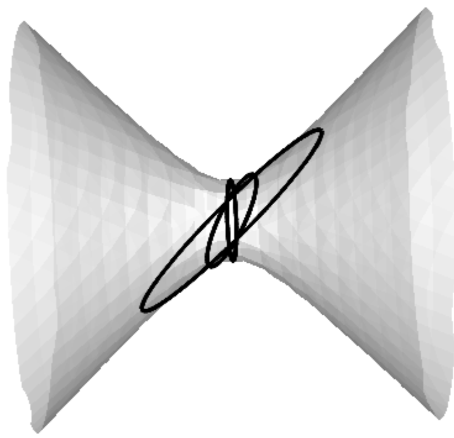

Figure 4.4: Some geodesics on AdS_2

4.5.2 Dirac modes on Minkowski space

In [13], Loss and Yau used a particular formula to construct gauge potentials for a given spinor so that the spinor is a zero-mode of the Dirac operator coupled to the gauge potential. This formula has a simple Lorentzian analogue, namely

$$A_i = \frac{1}{|\vec{\Sigma}|}\left(\frac{1}{2}\varepsilon_i^{\ jk}\partial_j \Sigma_k + \mathrm{Im}(\Psi^\dagger \partial_i \Psi)\right), \tag{4.5.13}$$

where $\Sigma_i = 2i\Psi^\dagger \mathsf{t}_i \Psi$. However, its use is problematic because Lorentzian spinors may be null even when they are not vanishing. This result is Lemma A.3 in Appendix A

Our construction of magnetic modes proceeds differently. We use Lemma 4.7 to obtain magnetic Dirac modes on $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ directly from the vortex magnetic modes of Theorem 4.10 on $SU(1,1)$. The magnetic field in Minkwoski space is obtained from the magnetic field $F_{A'}$ on $SU(1,1)$ via pull-back with the inverse stereographic projection H. The magnetic field lines are therefore the images, under stereographic projection, of the fields lines on $SU(1,1)$. While the field lines on $SU(1,1)$ are all closed, they also leave the domain of the stereographic projection. As a result, the image curves in $\mathcal{I}_\ell$ are not closed. Instead, they are of the form shown in Fig. 4.5.

For explicit formulae on Minkowski space, it is convenient to work in vector notation where a one-form is expanded as $A = \vec{A} \cdot d\vec{x}$ on $\mathcal{I}_\ell$, and where magnetic two-forms are expressed in terms of vector fields according to (4.4.7). In particular, the inhomogeneous term in the equation (4.5.6) governing vortex magnetic modes pulls back to the two-form

$$-\frac{1}{4} H^*(\varsigma^1 \wedge \varsigma^2) = -\frac{4\ell^2}{(\ell^2 + r^2)^2} \star_{\mathbb{R}^{1,2}} \vartheta^0 = \frac{1}{2} \varepsilon_{ijk} b^i dx^j \wedge dx^k, \tag{4.5.14}$$

and the corresponding magnetic field is

$$\vec{b} = \frac{-4\ell^2}{(\ell^2 + r^2)^3} \begin{pmatrix} \ell^2 - r^2 + 2(x^0)^2 \\ 2(x^2 \ell - x^1 x^0) \\ -2(x^1 \ell + x^2 x^0) \end{pmatrix}. \tag{4.5.15}$$

The field lines of $\vec{b}$ are the fibres of the fibration $\rho : SU(1,1) \to H^2$, and plotted in Fig. 4.5.

Since vortex magnetic modes on $SU(1,1)$ satisfy a non-linear equation in addition to the linear Dirac equation, we expect the same to be true for the vortex magnetic modes on Minkowski space. We define them as follows.

Definition 4.11

A pair (Ψ, A) *of a spinor* Ψ *and a one-form* $A = \vec{A} \cdot d\vec{x}$ *on* $\mathbb{R}^{1,2}$ *is called a vortex*

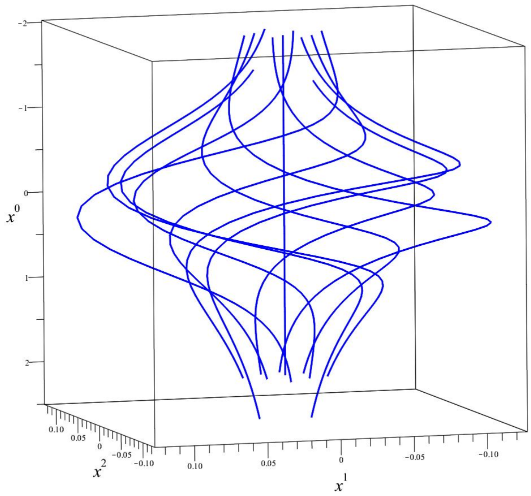

Figure 4.5: The magnetic field lines for the pull-back of vortex magnetic modes to Minkowski space. In particular, they are the magnetic fields lines of the background field $\vec{b}$. They are also the images of the fibres illustrated in Fig. 4.4 under the stereographic projection.

magnetic mode in Minkowski space if it satisfies the coupled equations

$$\not{D}_{\mathbb{R}^{1,2},A}\Psi = 0, \quad \vec{B} = -2i\Psi^\dagger\vec{\mathfrak{t}}\Psi + \vec{b}, \tag{4.5.16}$$

where $\vec{B} = \nabla \times \vec{A}$ *and* $\vec{b}$ *is the background field given in* (4.5.15)

The coupled equations in this Definition formally resemble the dimensionally reduced Seiberg-Witten equations, perturbed by the background field $\vec{b}$. The role of the Seiberg-Witten equations in differential topology makes essential use of a Riemannian metric, and a Lorentzian version like the one defined here does not appear to have been studied.

Combining many of the results of this Chapter, we arrive at the following explicit construction of vortex magnetic modes on $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$:

Corollary 4.12

Any given bundle map $V : SU(1,1) \to SU(1,1)$ *covering a holomorphic map* $f : H^2 \to H^2$ *determines a smooth vortex magnetic mode on* $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$. *Explicitly, extracting the vortex configuration* (Φ, A) *on* $SU(1,1)$ *from* $\mathcal{A} = V^{-1}dV$ *via* (4.3.7), *the vortex magnetic mode is given by*

$$\Psi = G \begin{pmatrix} \Omega^{-1} H^* \Phi \\ 0 \end{pmatrix}, \quad A'_H = -H^*(A + \frac{3}{4}\varsigma^0). \tag{4.5.17}$$

Proof. The result follows by composing the construction of magnetic Dirac modes from vortex configurations with the construction of vortex configurations from bundle maps. We use Theorem 4.3 to construct a vortex configuration (Φ, A) on $SU(1,1)$ from the bundle map V, then Theorem 4.10 to construct a vortex magnetic mode (Ψ, A') on $SU(1,1)$ from (Φ, A). Finally Lemma 4.7 is used to pull it back to $\mathcal{I}_\ell$. The confirmation that the magnetic mode thus obtained satisfies the coupled equations (4.5.16) with gauge field and magnetic field

$$A'_H = \vec{A}'_H \cdot d\vec{x}, \qquad \vec{B}'_H = \nabla \times \vec{A}'_H, \tag{4.5.18}$$

is a straightforward calculation, which is analogous to the one carried out for the Euclidean version in Chapter 3. □

The Corollary allows one to construct solutions of the gauged Dirac equation and to solve initial value problems in Minkowski space. The restriction to $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ is not necessarily a problem in practice since ℓ can be chosen arbitrarily. By choosing it large enough, one can capture initial data on any bounded subset of a Cauchy surface.

Chapter 5

A unifying picture of vortex configurations

We are now in the position to provide a unifying picture of vortices in two dimensions, vortex configurations in three dimensions and, in certain cases, magnetic Dirac-modes in three dimensions. This Chapter is essentially a unification of the topics of Chapter 3 and Chapter 4 with the discussion extended to be applicable to all five of the (λ_0, λ) vortices considered in Chapter 2, see Table 5.1 for a reminder of these. We start with a general definition of vortex configurations on the group manifold $\mathbb{H}^1_{\lambda_0}$ and their relationship to flat connections. In the $\lambda_0 = \pm 1$ cases, where the vortex configurations are on $SU(2)$ or $SU(1,1)$ the results of Chapters 3 and 4 on magnetic Dirac-modes can be extended to give a construction of magnetic modes in terms of Ambjorn-Olsen and Bradlow vortex configurations on $SU(1,1)$. The summary Figure for this Chapter is Figure 5.1.

Vortex type	λ_0	λ
Hyperbolic	1	1
Popov	-1	-1
Jackiw-Pi	0	-1
Ambjorn-Olsen	1	-1
Bradlow	1	0

Table 5.1: This table summarises the values of λ and λ_0 which correspond to each of the different types of vortices. We remind the reader that $\lambda = -C$ and $\lambda_0 = -C_0$, where C, C_0 is the notation used in [1, 24].

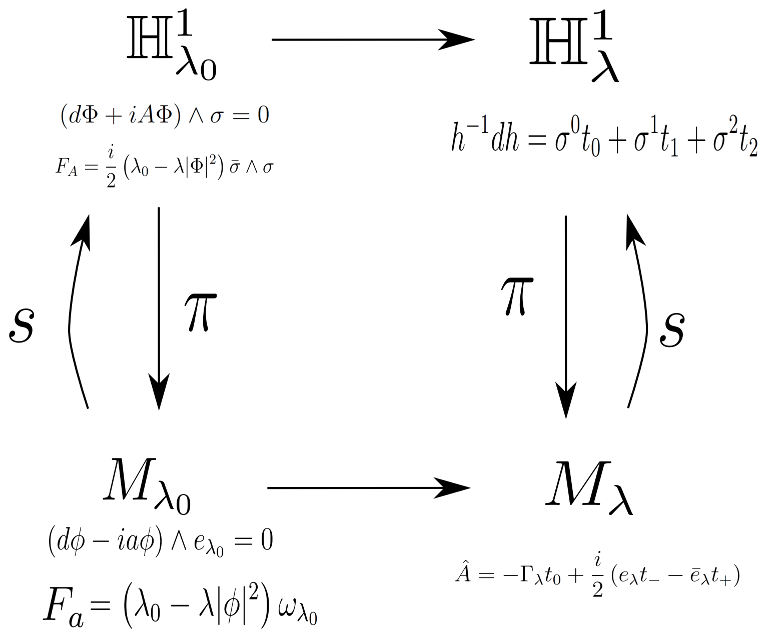

Figure 5.1: This summaries the four sets of equations and spaces that we relate in this Chapter. An extension on the left hand side relating the equations on $\mathbb{H}^1_{\lambda_0}$ to vortex magnetic modes on flat spaces can be included when $\lambda_0 \neq 0$.

5.1 The three dimensional story

5.1.1 Bundle structure

In Chapters 3 and 4 we made use of the circle bundles $U(1) \to SU(2) \to S^2$ and $U(1) \to SU(1,1) \to H^2$. Here we generalise this to the case of the surfaces M_λ discussed in Chapter 2. There it was mentioned that M_λ can be interpreted as a $U(1)$ quotient of $\mathbb{H}^1_\lambda$. This gives $\mathbb{H}^1_\lambda$ the structure of a circle bundle, $\pi : \mathbb{H}^1_\lambda \to M_\lambda$ with projection

$$\pi : h \mapsto z = \frac{z_2}{z_1}. \tag{5.1.1}$$

For $\lambda = -1$, $M_\lambda = \mathbb{C}P^1$ there is a local section $s : U \simeq \mathbb{C} \subset M_\lambda \to \mathbb{H}^1_\lambda$, where U is taken to exclude either the north or south pole. For $\lambda = 0, 1$ the surface is $M_\lambda = \mathbb{R}^2, H^2$ and there is a global section $s : M_\lambda \to \mathbb{H}^1_\lambda$.

In terms of the complex coordinates the section is

$$s : z \mapsto \frac{1}{\sqrt{1 - \lambda|z|^2}} \begin{pmatrix} 1 & \lambda\bar{z} \\ z & 1 \end{pmatrix}. \tag{5.1.2}$$

The section and projection give the relationship between the left-invariant one-forms $\sigma^0, \sigma, \bar{\sigma}$ on $\mathbb{H}^1_\lambda$, (2.2.18), and the complexified frame and spin connection, $e_\lambda, \bar{e}_\lambda, \Gamma_\lambda$ on M_λ, (2.3.11) and (2.3.13), in the following way:

$$s^*\sigma = ie_\lambda, \quad s^*\sigma^0 = -\Gamma_\lambda, \quad \pi^* e_\lambda = -i\frac{\bar{z}_1}{z_1}\sigma, \quad \pi^*\Gamma_\lambda = -\sigma^0 + id\ln\left(\frac{z_1}{\bar{z}_1}\right). \tag{5.1.3}$$

These are a generalisation of the relations given in the proof of Lemma 3.10 in Chapter 3. This relationship between left-invariant one-forms in three dimensions and the complexified basis in two dimensions leads to the following result

Proposition 5.1
Using the local section s, (5.1.2)*, the Cartan connection on M_λ, $\hat{A}$ from equation* (2.6.1)*, is trivialised as the pullback of the Maurer-Cartan one-form,* (2.2.14)*,*

$$\hat{A} = s^*\left(h^{-1}dh\right). \tag{5.1.4}$$

Proof. To prove this take the Maurer-Cartan one-form, (2.2.14), and use the first two expressions in (5.1.3) to interpret its pullback as $\hat{A}$. □

5.1.2 Vortex equations on $\mathbb{H}^1_{\lambda_0}$

The next step is to introduce the generalisation of a vortex configuration to $\mathbb{H}^1_\lambda$. Key ingredients are the equivariant functions on $\mathbb{H}^1_\lambda$ which are defined as follows.

Definition 5.2
The space of equivariant functions over $\mathbb{H}^1_\lambda$ is defined to be

$$C^\infty\left(\mathbb{H}^1_\lambda, \mathbb{C}\right)_N = \{F : \mathbb{H}^1_\lambda \to \mathbb{C} | 2iX_0 F = NF\}, \quad N \in \mathbb{Z}. \tag{5.1.5}$$

In Chapter 3 we had, following [33, 34], that $N \in \mathbb{N}^0$. That is because equivariant functions on S^3 are functions on the Lens space $S^3/\mathbb{Z}_N$ and are related to sections of the Hyperplane bundle. Here it is not a priori clear that we need to impose the same restriction to non-negative integers. In practice we only encounter equivariant functions constructed from holomorphic polynomials in z_1, z_2 and for these $N \in \mathbb{N}^0$.

A short computation using the local section s, (5.1.2), results in the following commutative diagram

$$\begin{tikzcd} C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})_N \arrow[r, "X_+"] \arrow[d, "s^*"'] & C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})_{N+2} \arrow[d, "s^*"] \\ C^\infty(M_\lambda) \arrow[r, "i(q\bar{\partial} - \lambda\frac{N}{2}z)"'] & C^\infty(M_\lambda) \end{tikzcd} \tag{5.1.6}$$

where $q = (1 - \lambda|z|^2)$. This result is included as Lemma B.1 in Appendix B. The fact that for $\lambda = 0$ the vector field X_+ is the lift of $\bar{\partial}$ is not particularly surprising as in this case $q = 1$, $X_+ = iz_1\bar{\partial}_2$ and the section s identifies z_2 with z.

In Chapters 3 and 4 there was an extra condition that $\lambda = \lambda_0$. This meant that when we interpreted vortex configurations as flat connections and trivialised in terms of a bundle map, the bundle map was from $\mathbb{H}^1_{\lambda_0}$ to itself. Here we do not have that $\lambda = \lambda_0$ and work with the two group manifolds, $\mathbb{H}^1_\lambda$ and $\mathbb{H}^1_{\lambda_0}$. This requires separate notation for the geometric objects on the two spaces. We use the notation s_a, σ^a, X_a for the generators, left-invariant one-forms and left-invariant vector fields respectively of the source group three manifold, $\mathbb{H}^1_{\lambda_0}$, and t_a, τ^a, Y_a for the same objects on the target group three manifold, $\mathbb{H}^1_\lambda$. Due to the details of our construction the Cartan connection is always valued in the Lie algebra of the target group. For $SU(2)$ and $SU(1,1)$ we follow the same conventions as Chapters 2 and 3 with the repeated warning that there are some sign differences with the conventions of Chapter 4.

Definition 5.3

Let A be a one form on $\mathbb{H}^1_{\lambda_0}$ and $\Phi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}$ a complex function. We call the pair (Φ, A) a vortex configuration if the vortex equations,

$$(d\Phi + iA\Phi) \wedge \sigma = 0 \qquad F_A = \frac{i}{2}\left(\lambda_0 - \lambda|\Phi|^2\right)\bar{\sigma} \wedge \sigma, \tag{5.1.7}$$

with $F_A = dA$, are satisfied.

As in Chapter 4 the equivariance conditions on A and Φ are not included in the definition. As was pointed out in Chapters 3 and 4 equivariance conditions, when A is normalised, follow from the vortex equations, (5.1.7) and Cartan's identity. They are

$$\mathcal{L}_{X_0}\Phi = -iA(X_0)\Phi, \tag{5.1.8}$$

$$\mathcal{L}_{X_0}A = dA(X_0). \tag{5.1.9}$$

As was observed in the $\lambda = \lambda_0 = \pm 1$ cases the vortex equations (5.1.7) possess the following $U(1)$ gauge invariance

$$(\Phi, A) \mapsto (e^{-i\beta}\Phi, A + d\beta), \qquad \beta \in C^\infty(\mathbb{H}^1_\lambda). \tag{5.1.10}$$

We learnt in Chapter 4 that $\mathbb{H}^1_{\lambda_0}$ is not simply connected when $\lambda_0 = 1$. This is also true for $\lambda_0 = 0$: $\mathbb{H}^1_0$ has the topology of $\mathbb{R}^2 \times S^1$. The generator of the fundamental group is the curve

$$\gamma = \{e^{\varphi t_0} \in \mathbb{H}^1_{\lambda_0} |\ \varphi \in [0, 4\pi)\}. \tag{5.1.11}$$

In the Euclidean case of Chapter 3, $\mathbb{H}^1_{-1} = S^3$, and since $\pi_1(S^3) = 0$ the equivalent curve to γ is contractible and we are able to trivialise the flat connection that we build from the vortex configuration in terms of a bundle map.

This is important for the main Theorem relating vortex configurations to flat connections where, as in Chapter 4, only flat connections with a prescribed holonomy around γ can be globally trivialised and thus give rise to global solutions of the vortex equations, (5.1.7).

Theorem 5.4 is a straightforward generalisation of Theorem 3.4 in Chapter 3 and Theorem 4.3 in Chapter 4 to the case of (λ_0, λ) vortex configurations.

Theorem 5.4
A vortex configuration on $\mathbb{H}^1_{\lambda_0}$ determines a gauge potential for a flat $\mathbb{H}^1_\lambda$ connection of the form

$$\mathcal{A} = \left(A + \sigma^0\right) t_0 + \frac{1}{2}\Phi\sigma t_- + \frac{1}{2}\bar{\Phi}\bar{\sigma}t_+. \tag{5.1.12}$$

Conversely, a flat $\mathbb{H}^1_\lambda$ connection $\mathcal{A}$ on $\mathbb{H}^1_{\lambda_0}$ such that

$$\mathcal{A}(X_0) = pt_0, \qquad \mathcal{A}(X_-) = \alpha t_0 + \Phi t_-, \tag{5.1.13}$$

for functions $p : \mathbb{H}^1_{\lambda_0} \to \mathbb{R}$ and $\alpha, \Phi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}$ determines a vortex configuration (Φ, A) through the expansion (5.1.12)*.*

Given $\mathcal{A}$, a gauge potential for a flat Lie$(\mathbb{H}^1_\lambda)$ connection on $\mathbb{H}^1_{\lambda_0}$ of the form (5.1.12) *which satisfies*

$$\int_\gamma \mathcal{A} = 2\pi n t_0, \tag{5.1.14}$$

for $n \in \mathbb{Z}$, can be trivialised as $U^{-1}dU$ for a bundle map $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ which covers a holomorphic map $f : M_{\lambda_0} \to M_\lambda$. Without loss of generality U can be taken to have the form

$$U : (z^1, z^2) \mapsto \frac{1}{\sqrt{|F_1|^2 - \lambda|F_2|^2}} \begin{pmatrix} F_1 & \lambda\bar{F}_2 \\ F_2 & \bar{F}_1 \end{pmatrix} \tag{5.1.15}$$

with $F_i : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}$ where $|F_1|^2 > \lambda|F_2|^2$.

The vortex configuration can be extracted from the bundle map as

$$\Phi\sigma = U^*\tau, \qquad A = U^*\tau^0 - \sigma^0. \tag{5.1.16}$$

Before giving the proof we should note that when $\lambda = 0, 1$ we can assume that $F_1 \neq 0$, since $|F_1|^2 > \lambda|F_2|^2$. However, in the $\lambda = -1$ case we can not make this assumption. This relates to the the fact that when $\lambda = -1$ the map $f = s^*\left(\frac{F_2}{F_1}\right)$ which U covers can have poles as a map $M_{\lambda_0} \to \mathbb{C}$ but is holomorphic as a map to $\mathbb{C}P^1$. This is exactly what was seen in Chapter 3, where f is a rational map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$.

The details of the proof are essentially the same as that of Theorem 3.4 in Chapter 3 and Theorem 4.3 in Chapter 4.

Proof. Given a $\mathbb{H}^1_\lambda$ connection on $\mathbb{H}^1_{\lambda_0}$ in the vortex gauge, (5.1.12), the flatness condition $d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$ is equivalent to

$$\left(d\left(\Phi\sigma\right) + i\left(A + \sigma^0\right)\Phi\right) = 0 \qquad dA = \frac{i}{2}\left(\lambda_0 - \lambda|\Phi|^2\right)\bar{\sigma} \wedge \sigma. \tag{5.1.17}$$

Using equation (2.2.17) these are seen to be equivalent to the vortex equations, (5.1.7).

For the converse expand the flat $\mathbb{H}^1_\lambda$ connection, $\mathcal{A}$, on $\mathbb{H}^1_{\lambda_0}$ in terms of the generators, t_0, t_+, t_-. The coefficients are linear combinations of σ^0, σ and $\bar{\sigma}$, as they form a basis of the cotangent space of $\mathbb{H}^1_{\lambda_0}$. Imposing the conditions in Equation (5.1.13) leads the gauge potential $\mathcal{A}$ being in the vortex form, Equation (5.1.12), with Higgs field Φ, and abelian gauge potential

$$A = (p-1)\,\sigma^0 + \frac{1}{2}\left(\alpha\sigma + \bar{\alpha}\bar{\sigma}\right). \tag{5.1.18}$$

The same calculation as above then gives that the vortex equations, (5.1.7), are satisfied.

To globally trivialise a flat connection $\mathcal{A}$ on $\mathbb{H}^1_{\lambda_0}$ in terms of $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ as $\mathcal{A} = U^{-1}dU$, its path-ordered exponential must be path independent. If this is the case U can be constructed explicitly from $\mathcal{P}\exp\left(\int_{\tilde{\gamma}} \mathcal{A}\right)$ along any path $\tilde{\gamma}$, starting at a fixed but arbitrary base point, [40].

As $\mathcal{A}$ is a flat connection the non-abelian Stokes theorem implies that the path-ordered exponential is path independent for contractible paths. The conditions in (5.1.13) ensure that the path-ordered exponential of $\mathcal{A}$ along γ coincides with the exponential of the ordinary integral. Then (5.1.14) implies that

$$\mathcal{P}\exp\left(\int_\gamma \mathcal{A}\right) = \mathbb{I}. \tag{5.1.19}$$

Flatness of the connection $\mathcal{A}$ and the non-abelian Stokes theorem then combine to give that the path-ordered exponential of $\mathcal{A}$ along any closed curve in $\mathbb{H}^1_{\lambda_0}$ is the identity. This gives the path independence of the path-ordered exponential.

The final part of the proof is to show that for $\mathcal{A} = U^{-1}dU$ satisfying (5.1.13) that U is a bundle map covering a holomorphic function $M_{\lambda_0} \to M_\lambda$. The first condition in (5.1.13) becomes

$$X_0 U = pUt_0, \tag{5.1.20}$$

with $p : \mathbb{H}^1_{\lambda_0} \to \mathbb{R}$. This is just the infinitesimal statement that U maps the fibres of $\mathbb{H}^1_{\lambda_0} \to M_{\lambda_0}$ to the fibres of $\mathbb{H}^1_\lambda \to M_\lambda$, in other words that U is a bundle map.

Complex conjugation of the second condition in (5.1.13) implies that

$$U^{-1}X_+U = \bar{\alpha}t_0 + \bar{\Phi}t_+. \tag{5.1.21}$$

Now apply

$$U^{-1}dU = U^*\tau^0 t_0 + \frac{1}{2}U^*\tau t_- + \frac{1}{2}U^*\bar{\tau}t_+ \tag{5.1.22}$$

to X_+, the condition in (5.1.21) is thus equivalent to

$$U^*\tau(X_+) = 0. \tag{5.1.23}$$

We now need to show that this is equivalent to U covering a holomorphic map.

Using the parameterisation of U in terms of the functions F_i defined in (5.1.15) we see that Equation (5.1.20) becomes

$$X_0\left(\frac{F_i}{\sqrt{|F_1|^2-\lambda|F_2|^2}}\right) = \frac{i}{2}p\left(\frac{F_i}{\sqrt{|F_1|^2-\lambda|F_2|^2}}\right). \tag{5.1.24}$$

From this it follows that the map $\pi \circ U = \frac{F_2}{F_1}$ has equivariant degree zero, from Definition 5.2, and that U covers

$$f = s^*\left(\frac{F_2}{F_1}\right) : M_{\lambda_0} \to M_\lambda. \tag{5.1.25}$$

Using (5.1.6) for $\frac{F_2}{F_1}$ we find that f being holomorphic is equivalent to

$$X_+\left(\frac{F_2}{F_1}\right) = 0. \tag{5.1.26}$$

Returning to (5.1.23) use (2.2.18) to see that

$$U^*\tau(X_+) = \frac{2i}{|F_1|^2-\lambda|F_2|^2}\left(F_1X_+F_2 - F_2X_+F_1\right) = \frac{2i}{|F_1|^2-\lambda|F_2|^2}F_1^2X_+\frac{F_2}{F_1}, \tag{5.1.27}$$

with the last equality holding away from the zeros of F_1. Thus the condition (5.1.23) is equivalent to $f = s^*\left(\frac{F_2}{F_1}\right)$ being holomorphic away from the zeros of F_1. This means that for $\lambda \neq -1$ the result has been established. For the $\lambda = -1$ case f will have poles at the zeros of F_1 and is a holomorphic map $M_{\lambda_0} \to S^2$. □

The $U(1)$ gauge invariance of the vortex equations that we saw in (5.1.10) is implemented at the level of the bundle map U as

$$U \mapsto \tilde{U} = U e^{\beta t_0}, \quad \beta \in C^\infty(\mathbb{H}^1_{\lambda_0}). \tag{5.1.28}$$

This gives a new trivialisation which defines the same f as U with the connection in this new trivialisation $\tilde{U}^{-1} d\tilde{U}$ differing from $\mathcal{A} = U^{-1}dU$ by the gauge transformation in (5.1.10).

Guided by the trivial lift in the $\lambda = \lambda_0 = 1$ case, (4.3.23), we observe that when $\lambda \neq -1$ a vortex configuration can be constructed from a given holomorphic map $f : M_{\lambda_0} \to M_\lambda$ by the choice

$$F_1(z_1, z_2) = 1, \quad F_2(z_1, z_2) = f\left(\frac{z_2}{z_1}\right). \tag{5.1.29}$$

This trivial lift results in a connection $\mathcal{A}$ that is constant along the fibres since $\mathcal{A}(X_0) = 0$ which along with flatness gives $\mathcal{L}_{X_0}\mathcal{A} = 0$. The trivial lift is a direct consequence of the bundles being trivial in the $\mathbb{H}^1_1$ and $\mathbb{H}^1_0$ cases.

5.1.3 The case of finite equivariant degree

So far we have not placed any restrictions on Φ having finite equivariant degree, in the sense of Definition 5.2. From Chapters 3 and 4 we know that these vortex configurations are particularly interesting as the equivariant degree of Φ can be related to the vortex number of the Higgs field, ϕ, of a vortex in two dimensions. In particular in Chapter 4 we saw how to construct a bundle map which is a natural lift of a finite Blaschke product. This enabled us to lift hyperbolic vortices on H^2 to vortex configurations on $SU(1,1)$ in a non-trivial manner. A key piece of this construction was the explicit form of the map $f : H^2 \to H^2$ for a $N-1$ vortex given by the Blaschke product

$$f = \prod_{k=1}^{N} \frac{z - c_k}{1 - \bar{c}_k z} = \frac{f_2}{f_1}. \tag{5.1.30}$$

The lift is then to take $\frac{f_2}{f_1} = \frac{F_2}{F_1}$. There is a very similar story for Popov vortices which we saw in Chapter 3. It is known from [1, 11] that the Popov vortex equations are solved by rational maps $f = \frac{p}{q} : S^2 \to S^2$, with p, q polynomials. The non-trivial lift then amounts to finding two complex polynomials on $SU(2)$ whose ration is $\frac{p}{q}$. We refer to this lift as being non-trivial, in contrast to the trivial lift in (5.1.29), since the functions F_i have a non-trivial equivariant degree. In fact the equivariant degree is related to the vortex number. The equivariant degree of the Higgs field, Φ, is always $2N-2$ in the sense of Definition 5.2 while the vortex number is different for the different types of vortex.

To be able to exhibit a similar non-trivial lift for the Bradlow, Jackiw-Pi or Ambjorn-Olsen vortices would require similar explicit expressions for the map f.

Jackiw-Pi vortices

In [41] it was shown that the Jackiw-Pi vortex equations on $\mathbb{R}^2$ with a finite number of zeros are solved by a rational map $f = \frac{p}{q} : \mathbb{R}^2 \to S^2$ with $\deg(p) < \deg(q)$. For example a $2N$-vortex solution is given by

$$p(z) = \sum_{i=0}^{M} a_i z^i, \qquad q(z) = \sum_{i=0}^{N} b_i z^i, \quad M < N, \tag{5.1.31}$$

with the understanding that p and q have no common factors, at least one of a_0, b_0 are non-zero and $b_N \neq 0$. In this case we can write down the following homogeneous polynomials

$$P(z_1, z_2) = \sum_{i=0}^{M} a_i z_1^{N-i} z_2^i, \qquad Q(z_1, z_2) = \sum_{i=0}^{N} b_i z_1^{N-i} z_2^i, \tag{5.1.32}$$

which satisfy

$$s^* \left(\frac{P}{Q} \right) = \frac{p}{q}. \tag{5.1.33}$$

Examples of Jackiw-Pi vortices with $N = 1$ and $N = 2$, including plots of $|\phi|^2$, are given in [42, 43].

The following Corollary of Theorem 5.4 explains how to construct a $(0, -1)$ vortex configuration from a Jackiw-Pi vortex.

Corollary 5.5

Let $U : \mathbb{H}^1_0 \to SU(2)$ be the bundle map (5.1.15) with

$$F_1(z_1, z_2) = Q(z_1, z_2), \qquad F_2(z_1, z_2) = P(z_1, z_2), \tag{5.1.34}$$

with P, Q as given in (5.1.32). The vortex configuration (Φ, A) constructed from the connection $\mathcal{A} = U^{-1}dU$ through Theorem 5.4 has a gauge field which satisfies the normalisation condition

$$A(X_0) = N - 1, \tag{5.1.35}$$

and a Higgs field of equivariant degree $2N - 2$. In terms of P, Q the vortex configuration is expressed as

$$\Phi = \frac{Q\partial_2 P - P\partial_2 Q}{z_1\left(|P|^2 + |Q|^2\right)}, \tag{5.1.36}$$

and

$$A = (N-1)\,\sigma^0 + \frac{i}{2}X_- \ln D^2\sigma - \frac{i}{2}X_+ \ln D^2\bar{\sigma}. \tag{5.1.37}$$

with $D^2 = |P|^2 + |Q|^2$.

Proof. From the explicit form of F_1, F_2 we have that

$$X_0 U = NUt_0, \tag{5.1.38}$$

and thus $\mathcal{A}(X_0) = Nt_0$. Then using (5.1.12) we have that $A(X_0) = N - 1$ and by (5.1.8) we have that

$$\mathcal{L}_{X_0}\Phi = -i(N-1)\Phi, \tag{5.1.39}$$

so the Higgs field has equivariant degree $2N - 2$. The explicit expression for the Higgs field follows from

$$\Phi = \frac{1}{2}\left(U^*\tau\right)(X_-). \tag{5.1.40}$$

The expression for A follows from (5.1.16) and computing

$$\left(U^*\tau^0\right)(X_0) = N, \quad \left(U^*\tau^0\right)(X_+) = -iX_+ \ln D^2, \quad \left(U^*\tau^0\right)(X_-) = iX_- \ln D^2. \tag{5.1.41}$$

$\square$

In [1] the case of Jackiw-Pi vortices on the torus is discussed, there the map f is a doubly periodic elliptic function. As a vortex on the torus it has a finite vortex number, $2N$ where N is the number of poles of f. However, as a vortex on $\mathbb{R}^2$ it has an infinite number of zeros. The torus is obtained from $\mathbb{R}^2$ by quotienting with discrete subgroup of SE_2 and demanding that the zeros of the Higgs field on $\mathbb{R}^2$ are periodic under this subgroup and there are $2N$ of them in the principal domain. The only way to lift these vortices seems to be via the trivial lift (5.1.29).

The most popular example of a Jackiw-Pi vortex on $\mathbb{R}^2$ [41–43] is to take

$$f = \frac{1}{z^N}. \tag{5.1.42}$$

For this choice of f the F_i are given by

$$F_1 = z_2^N, \qquad F_2 = z_1^N. \tag{5.1.43}$$

For this vortex the Higgs field of the vortex configuration is given by

$$\Phi = -N\frac{z_1^{N-1}z_2^{N-1}}{|z_1|^{2N} + |z_2|^{2N}}. \tag{5.1.44}$$

This can be explicitly seen to satisfy $2iX_0\Phi = (2N-2)\Phi$ and thus Φ is a degree $2N-2$ equivariant function.

Ambjorn-Olsen Vortices

From [11] we know that Ambjorn-Olsen vortices are constructed from a holomorphic map $f : H^2 \to S^2$, subject to $|f(z)| \to 1$ as $|z| \to 1$. These maps can be expressed in terms of their m zeros, $c_1, \cdots, c_m$, and n poles, $d_1, \cdots, d_n$, as

$$f(z) = \frac{f_2}{f_1} = \prod_{i=1}^{m}\left(\frac{z - c_i}{1 - \bar{c}_i z}\right)\prod_{j=1}^{n}\left(\frac{1 - \bar{d}_j z}{z - d_j}\right). \tag{5.1.45}$$

To see this we follow [44] and use that the zeros and poles of f define the Blaschke products

$$f_2 = \prod_{i=1}^{m}\left(\frac{z - c_i}{1 - \bar{c}_i z}\right), \quad f_1 = \prod_{j=1}^{n}\left(\frac{z - d_j}{1 - \bar{d}_j z}\right). \tag{5.1.46}$$

The ratio of these Blaschke products $\frac{f_2}{f_1}$ has the same zeros and poles as f and their ratio $\frac{ff_1}{f_2}$ is a holomorphic function with no zeros and no poles satisfying $|f(z)| = 1$ for $|z| = 1$. Liouville's theorem then gives that this ratio is a constant, $\mu \in \mathbb{C}$ such that $|\mu| = 1$, multiplying f by a constant does not change the vortex that we construct from f so we can take $\mu = 1$.

This rational expression for f leads us to another example of a non-trivial lift of a vortex to a $(1, -1)$ vortex configuration.

Corollary 5.6

Let $U : \mathbb{H}^1_1 \to SU(2)$ be the bundle map (5.1.15) with

$$F_1(z_1, z_2) = \prod_{i=1}^{n}\prod_{j=1}^{m} (z_1 - \bar{c}_i z_2)(z_2 - d_j z_1), \quad F_2(z_1, z_2) = \prod_{i=1}^{n}\prod_{j=1}^{m} (z_2 - c_i z_1)\left(z_1 - \bar{d}_j z_2\right). \tag{5.1.47}$$

The vortex configuration (Φ, A) constructed from the connection $\mathcal{A} = U^{-1}dU$ through Theorem 5.4 has a gauge field which satisfies the normalisation condition

$$A(X_0) = N - 1, \tag{5.1.48}$$

and a Higgs field of equivariant degree $2N - 2$, for $N = n + m$. In terms of F_1, F_2 the vortex configuration is expressed as

$$\Phi = \frac{F_1 \partial_2 F_2 - F_2 \partial_2 F_1}{z_1 \left(|F_1|^2 + |F_2|^2\right)}, \tag{5.1.49}$$

and

$$A = (N - 1)\,\sigma^0 + \frac{i}{2} X_- \ln D^2 \sigma - \frac{i}{2} X_+ \ln D^2 \bar{\sigma}. \tag{5.1.50}$$

with $D^2 = |F_1|^2 + |F_2|^2$.

The proof is exactly the same as that of Corollary 5.5 and uses the explicit expressions for the maps F_i, (5.1.47), along with the identification $n + m = N$.

Given a method of writing the holomorphic map used to construct a Bradlow vortex as a rational function the same method as used here give rise to a non-trivial lift in that case as well.

5.1.4 Lifting vortices to vortex configurations

In Chapters 3 and 4 we saw how to lift Popov and hyperbolic vortices to vortex configurations on $SU(2)$ and $SU(1,1)$, while in the previous Section we saw one way to lift Jackiw-Pi vortices to vortex configurations on $\mathbb{H}^1_0$ and Ambjorn-Olsen vortices on $\mathbb{H}^1_1$. Now we see how to lift (λ_0, λ) vortices to vortex configurations on $\mathbb{H}^1_{\lambda_0}$. We have already seen in (5.1.29) that, when both circle bundles are trivial, a vortex configuration can be constructed directly from a vortex. A natural setting to state the relationship between vortices and vortex configurations is at the level of the flat connections. The first step is for a function $g : M_{\lambda_0} \to \mathbb{C}$ define the map

$$f_g : M_{\lambda_0} \backslash \{q_i\} \to \mathbb{H}^1_\lambda, \quad r_g = \begin{pmatrix} \frac{\bar{g}}{|g|} & 0 \\ 0 & \frac{g}{|g|} \end{pmatrix}, \tag{5.1.51}$$

where the q_i are the zeros of g. Armed with Proposition 5.1 the following result spells out the relationship between the flat connections for a vortex on M_{λ_0} and for a vortex configuration on $\mathbb{H}^1_{\lambda_0}$.

Lemma 5.7

For U the bundle map from (5.1.15) *covering the holomorphic map f, the gauge potential $f^*\hat{A}$, given in* (2.6.4)*, of a vortex on M_{λ_0} is related to $\mathcal{A} = U^{-1}dU$ through the, possibly singular, gauge transformation r_{f_1}, where $f_1 = F_1 \circ s$:*

$$f^*\hat{A} = r_{f_1}^{-1} s^* (\mathcal{A})\, r_{f_1} + r_{f_1}^{-1} dr_{f_1}. \tag{5.1.52}$$

Proof. A direct computation shows that $s \circ f = (U \circ s)\, r_{f_1}$ then using that $\mathcal{A} = U^{-1}dU$ and that $\hat{A} = s^* (h^{-1}dh)$, from Proposition 5.1. $\square$

As was observed in Chapter 4 picking the gauge where the lift is given by (5.1.29) results in $r_{f_1} = \mathbb{I}$ and the two connections line up exactly. We mentioned above that r_{f_1} may be singular in some cases, this was encountered explicitly in Chapter 3. Singularities occur when F_1 has zeros, this happens when $\lambda = -1$. The case of Popov vortices was dealt with in detail in Chapter 3 and it should be noted that for Jackiw-Pi vortices the same careful approach to dealing with the singularities is needed.

We do not present here the global Cartan geometry picture. However, note that as we saw in Chapter 3, the standard method needs to be extended to include singularities in the cases with $\lambda = -1$.

5.2 Dirac operators on $\mathbb{H}^1_{\lambda_0}$ and its Lie algebra

5.2.1 The Lie algebra of $\mathbb{H}^1_{\lambda_0}$

In Chapters 3 and 4 we saw how to relate the Dirac operators on $SU(2)$ and $SU(1,1)$ to the Dirac operators on $\mathbb{R}^3$ and $\mathbb{R}^{1,2}$. Now we give a unified interpretation of those results which can be extended to include vortex magnetic modes from Bradlow and Ambjorn-Olsen vortices. We do not construct vortex magnetic modes on $\mathbb{H}^1_0$ from Jackiw-Pi vortices due to the metric being singular. We present unified notation to convert and extend the results of the previous chapters to the setting of $\mathbb{H}^1_{\lambda_0}$. There are minimal proofs in this subsection as when $\lambda = -1$ the results are just those stated in Chapter 3 and when $\lambda = 1$ they are closely related to the results of Chapter 4, but in the alternative conventions adopted throughout the rest of the thesis.

We stress that the results of this section are all under the assumption that $\lambda_0 \neq 0$.

The Killing metric on the Lie algebra of $\mathbb{H}^1_{\lambda_0}$ depends on λ_0 and is

$$ds^2_{\lambda_0} = -\frac{1}{\lambda_0}\left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2, \tag{5.2.1}$$

in particular this metric would be singular if the $\lambda_0 = 0$ case was included[1]. We are assuming the oriented (pseudo) orthonormal frame

$$\left(dx^0, dx^1, dx^2\right), \tag{5.2.2}$$

so that the volume form is

$$dx^0 \wedge dx^1 \wedge dx^2 \tag{5.2.3}$$

[1]This is similar to the metric

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2$$

with $\frac{1}{\lambda_0}$ playing the role of c^2. The $\lambda_0 \to 0$ limit is then the Newtonian limit of the relativistic, $\lambda_0 = 1$, case.

A point $\vec{x} \in \mathbb{R}^3_{\lambda_0}$ is given by $\vec{x} = x^a \vec{b}_a$ with $\vec{b}_a$ an oriented basis such that $g(\vec{b}_a, \vec{b}_b) = g_{ab}$, for g as in (2.2.1). For $\vec{x}, \vec{y} \in \mathbb{R}^3_{\lambda_0}$ the scalar product is given by

$$\vec{x} \cdot \vec{y} = g_{ab} x^a y^b, \tag{5.2.4}$$

and the distance to $\vec{x}$ is

$$r^2 = \vec{x} \cdot \vec{x} = -\lambda_0 (x_0)^2 + (x_1)^2 + (x_2)^2. \tag{5.2.5}$$

The cross product of $\vec{x}, \vec{y} \in \mathbb{R}^3_{\lambda_0}$ is

$$\vec{x} \times \vec{y} = \varepsilon_{ij}{}^k x^i y^j b_k, \tag{5.2.6}$$

with $\varepsilon_{012} = 1, \varepsilon^{012} = -\lambda_0$.

We call this space

$$\mathbb{R}^3_{\lambda_0} = \left(\mathbb{R}^3, ds^2_{\lambda_0}\right). \tag{5.2.7}$$

On $\mathbb{H}^1_{\lambda_0}$ we work with the oriented, orthonormal frame

$$\left(\frac{1}{2}\sigma^0, \frac{1}{2}\sigma^1, \frac{1}{2}\sigma^2\right), \tag{5.2.8}$$

with respect to which the bi-invariant metric is

$$ds^2 = \frac{1}{4}\left(-\frac{1}{\lambda_0}\left(\sigma^0\right)^2 + \left(\sigma^1\right)^2 + \left(\sigma^2\right)^2\right), \tag{5.2.9}$$

and the orientation[2] is

$$\mathrm{Vol}_{\mathbb{H}^1_{\lambda_0}} = \frac{1}{8}\sigma^1 \wedge \sigma^0 \wedge \sigma^2. \tag{5.2.10}$$

We construct two maps between $\mathbb{R}^3_{\lambda_0}$ and $\mathbb{H}^1_{\lambda_0}$,

$$G, H : \mathbb{R}^3_{\lambda_0} \to \mathbb{H}^1_{\lambda_0}. \tag{5.2.11}$$

In Chapters 3 and 4 we saw that H is a scaled version of inverse stereographic projection and G is the inverse gnomonic projection with the two maps related

[2]The slightly unconventional ordering is so that we make contact with the volume form on $\mathbb{R}^3$ after stereographic projection in the $\lambda = 1, -1$ cases.

through $H(\vec{x}) = G(\vec{x})^2$. It was also pointed out in Chapter 4 that G, H are not maps from all of $\mathbb{R}^3_{\lambda_0}$ to $\mathbb{H}^1_{\lambda_0}$ but only from the subspace $\mathcal{I} \subset \mathbb{R}^3_{\lambda_0}$ defined as

$$\mathcal{I} = \{(x^0, x^1, x^2) \in \mathbb{R}^3_{\lambda_0} | \lambda_0 r^2 < 1\}. \tag{5.2.12}$$

In the $SU(2)$ ($\lambda_0 = -1$) case, $\mathcal{I} = \mathbb{R}^3$ since the above condition becomes $r^2 > -1$. However, in the $SU(1,1)$ ($\lambda_0 = 1$) case, the condition on r^2 is that

$$r^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 < 1, \tag{5.2.13}$$

and we saw in Chapter 4 that $\mathcal{I}$ is the interior of a single sheeted hyperboloid.

In this notation we have that H is

$$H : \mathcal{I} \to \mathbb{H}^1_{\lambda_0}, \tag{5.2.14}$$

$$\vec{x} \mapsto \frac{1+\lambda_0 r^2}{1-\lambda_0 r^2}\mathbb{I} - \frac{4}{1-\lambda_0 r^2}\vec{x}\cdot\vec{t} = \frac{1}{1-\lambda_0 r^2}\begin{pmatrix} 1+\lambda_0 r^2 + 2ix^0 & -2i\lambda_0\left(x^1 - ix^2\right) \\ 2i\left(x^1 + ix^2\right) & 1+\lambda_0 r^2 - 2ix^0 \end{pmatrix}. \tag{5.2.15}$$

The analogue of the inverse gnomonic projection[3] is

$$G : \mathcal{I} \to \mathbb{H}^1_{\lambda_0}, \qquad \vec{x} \mapsto \frac{\mathbb{I} - 2\vec{x}\cdot\vec{t}}{\sqrt{1-\lambda_0 r^2}}. \tag{5.2.16}$$

Pulling back the left-invariant one-forms, σ^a, with H we define a co-frame on $\mathbb{R}^3_{\lambda_0}$, ϑ^a, as

$$H^*\sigma^a = -\frac{1}{\Omega}\vartheta^a, \tag{5.2.17}$$

where the scale factor Ω is defined as

$$\Omega = \frac{1-\lambda_0 r^2}{4}. \tag{5.2.18}$$

[3]It is interesting to observe that we can make sense of both H and G in the $\lambda_0 = 0$ case. Key to this is that by keeping track of factors of λ_0 we find that $-\lambda_0 r^2 = (x^0)^2$.

Computing $H^{-1}dH = H^* \left(h^{-1}dh\right)$ we then find that

$$\begin{aligned}\vec{\vartheta}\cdot\vec{t} &= \frac{1}{1-\lambda_0 r^2}\left(\left(1+\lambda_0 r^2\right)\left(d\vec{x}\cdot\vec{t}\right) - 2\lambda_0\left(d\vec{x}\cdot\vec{x}\right)\left(\vec{x}\cdot\vec{t}\right) + 2\left(\vec{x}\times d\vec{x}\right)\cdot\vec{t}\right), \\ &= G^{-1}\left(d\vec{x}\cdot\vec{t}\right)G. \end{aligned} \tag{5.2.19}$$

This means that the ϑ^a give a rotated basis for the co-tangent space, with the rotation given by G acting in the adjoint representation.

Lemma 5.8

The pull-backs of the Maurer-Cartan one-form on $\mathbb{H}^1_{\lambda_0}$ by the maps G, H are related via

$$H^{-1}dH = G^{-1}dG + G^{-1}(G^{-1}dG)G, \tag{5.2.20}$$

with the inverse relation

$$G^{-1}dG = \frac{1}{2}H^{-1}dH - \lambda_0 \star\left(d\Omega\wedge H^{-1}dH\right). \tag{5.2.21}$$

Here $\star$ is the Hodge star operator on $\mathbb{H}^1_\lambda$ with respect to the orientation (5.2.10).

As stated at the start of this Section the proof of this result is omitted as it has already been proved in the $\lambda_0 = \pm 1$ cases separately in Chapters 3 and 4.

5.2.2 Dirac operators

We now construct the Dirac operators on $\mathbb{H}^1_{\lambda_0}$ and $\mathbb{R}^3_{\lambda_0}$. The spin connection on $\mathbb{H}^1_{\lambda_0}$ is

$$\Gamma_{\mathbb{H}^1_{\lambda_0}} = -\frac{1}{8}[\gamma_a,\gamma_b]\omega^{ab} = \frac{1}{2}h^{-1}dh. \tag{5.2.22}$$

Introducing θ, which is 1 in the Lorentzian case and i in the Euclidean case, such that $\theta^2 = \lambda_0$ the Dirac operator is written as

$$\not{D}_{\mathbb{H}^1_{\lambda_0}} = 4i\theta t^a X_a + \frac{3}{2}i\lambda_0\theta\mathbb{I}, \tag{5.2.23}$$

$$= -2\theta\begin{pmatrix}\lambda_0 X_0 & \lambda_0 X_- \\ -X_+ & -\lambda_0 X_0\end{pmatrix} + \frac{3}{2}i\lambda_0\theta\mathbb{I}. \tag{5.2.24}$$

The Dirac operator can be minimally coupled to an Abelian gauge potential A as

$$\not{D}_{\mathbb{H}^1_{\lambda_0},A} = 4i\theta t^a \left(X_a + iA_a\right) + \frac{3}{2}i\lambda_0\theta\mathbb{I} = -2\theta \begin{pmatrix} \lambda_0\left(X_0 + iA_0\right) & \lambda_0\left(X_- + iA_-\right) \\ -\left(X_+ + iA_+\right) & -\lambda_0\left(X_0 + iA_0\right) \end{pmatrix} + \frac{3}{2}i\lambda_0\theta\mathbb{I}. \tag{5.2.25}$$

The Dirac operator on $\mathbb{R}^3_{\lambda_0}$ minimally coupled to the gauge potential $\vec{A}\cdot\vec{dx}$ is

$$\not{D}_{\mathbb{R}^3_{\lambda_0},A} = 2i\theta t^a \left(\partial_a + iA_a\right) \tag{5.2.26}$$

Definition 5.9

A spinor $\Psi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}^2$ *that satisfies*

$$\not{D}_{\mathbb{H}^1_{\lambda_0},A}\Psi = 0 \tag{5.2.27}$$

is called a magnetic mode or magnetic Dirac mode of the Dirac operator $\not{D}_{\mathbb{H}^1_{\lambda_0}}$*, coupled to the connection* A*.*

Lemma 5.10

If $\Psi : \mathbb{H}^1_{\lambda_0} \to \mathbb{C}^2$ *is a magnetic Dirac mode of the Dirac operator* (5.2.25) *on* $\mathbb{H}^1_{\lambda_0}$ *coupled to the* $U(1)$ *gauge field* A*, then*

$$\Psi_H = G\Omega^{-1}H^*\Psi \tag{5.2.28}$$

is a zero-mode of the Dirac operator $\not{D}_{\mathbb{R}^3_{\lambda_0},H^*A}$ *on Euclidean 3-space coupled to the connection* H^*A*.*

5.2.3 Vortex magnetic modes

We are now ready to present the general discussion of vortex magnetic modes. These results are also unifications of the earlier results in Chapters 3 and 4. However, we are now able to construct vortex magnetic modes from Bradlow and Ambjorn-Olsen vortex configurations.

Definition 5.11

A pair (Ψ, A) *of a spinor* Ψ *and a one-form* A *on* $\mathbb{H}^1_{\lambda_0}$ *is said to be a vortex magnetic* *mode of the Dirac equation on* $\mathbb{H}^1_{\lambda_0}$ *if*

$$\not{D}_{\mathbb{H}^1_{\lambda_0},A}\Psi = 0, \quad F_A = -\lambda\frac{4i}{\ell} \star \Psi^\dagger h^{-1} dh\Psi - \lambda_0\frac{1}{4}\sigma^1 \wedge \sigma^2, \tag{5.2.29}$$

with $\star$ *the Hodge star operator on* $SU(1,1)$ *with respect to the metric* (5.2.9) *and orientation* (5.2.10).

Theorem 5.12

Given a vortex configuration (Φ, A) *on* $\mathbb{H}^1_{\lambda_0}$ *the pair*

$$\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \qquad A' = A + \frac{3}{4}\sigma^0, \tag{5.2.30}$$

is a vortex zero mode on $\mathbb{H}^1_{\lambda_0}$.

In the case of $\lambda = \lambda_0 = -1$ this becomes Theorem 3.7 in Chapter 3. While for $\lambda_0 = 1 = \lambda$ it becomes Theorem 4.10 in Chapter 4. The remaining two cases, $\lambda_0 = 1, \lambda = -1$ and $\lambda_0 = 1, \lambda = 0$ have the same proof as that for Theorem 4.10 but with the curvature term being

$$F_{A'} = F_A + \frac{3}{4}\lambda_0\sigma^1 \wedge \sigma^2 = \left(\lambda|\Phi|^2 - \frac{1}{4}\lambda_0\right)\sigma^1 \wedge \sigma^2. \tag{5.2.31}$$

5.2.4 Vortex magnetic modes on flat space

Combining with Lemma 5.10 vortex magnetic modes on $\mathbb{H}^1_{\lambda_0}$ can be converted in to magnetic modes on $\mathbb{R}^3_{\lambda_0}$.

Before stating what vortex magnetic modes pull back to on $\mathbb{R}^3_{\lambda_0}$ we need to establish what happens to the inhomogeneous term in (5.2.29). Computing its pullback we find

$$\frac{1}{4}H^*\left(\sigma^1 \wedge \sigma^2\right) = \frac{4}{\left(1 - \lambda_0 r^2\right)^2} \star_{\mathbb{R}^3_{\lambda_0}} \vartheta^0 = \frac{1}{2}\varepsilon^a{}_{bc}b_a dx^b \wedge dx^c. \tag{5.2.32}$$

The corresponding magnetic field

$$\vec{b} = \frac{1}{1-\lambda_0 r^2}\begin{pmatrix} 1+\lambda_0 r^2 + 2x_0^2 \\ 2\left(\lambda_0 x_2 + x_1 x_0\right) \\ -2\left(\lambda_0 x_1 - x_2 x_0\right) \end{pmatrix}, \tag{5.2.33}$$

is a background magnetic field, with field lines the fibres of the fibration $\pi : \mathbb{H}^1_{\lambda_0} \to M_{\lambda_0}$.

Examples of the field lines of this magnetic field were plotted in Figure 3.3 for $\lambda_0 = -1$ and in Figure 4.5 for $\lambda_0 = 1$. As we have excluded the $\lambda_0 = 0$ case the two cases already addressed capture the key structure of the field lines for all of the vortex magnetic modes. The only differences between the three types of vortex magnetic modes on $\mathbb{R}^3_1 = \mathbb{R}^{1,2}$ comes from the different coefficients of $\Psi^\dagger h^{-1} dh \Psi$ in (5.2.29).

The following definition of vortex magnetic modes extends those given in Chapters 3 and 4 to include the case of Bradlow and Ambjorn-Olsen vortex modes on $\mathbb{R}^3_1$.

Definition 5.13
A pair (Ψ, A) of a spinor Ψ and a one form $A = \vec{A}\cdot d\vec{x}$ is called a vortex magnetic mode on $\mathbb{R}^3_{\lambda_0}$ if the coupled equations

$$\not{D}_{\mathbb{R}^3_{\lambda_0}, H^*A}\Psi = 0, \qquad \vec{B} = -2i\lambda \Psi_H^\dagger \vec{t}\,\Psi_H - \lambda_0 \vec{b}, \tag{5.2.34}$$

where $\vec{B} = \nabla \times \vec{A}$ and $\vec{b}$ is the background field (5.2.33)*, are satisfied.*

The means of constructing examples of such vortex magnetic modes is through the following Corollary of our earlier results. The most natural way to state it is in the same manner as Corollary 4.12 from Chapter 4. This starts from a bundle map and results in explicit expressions for the spinor and gauge potential in terms of a vortex configuration.

Corollary 5.14
A bundle map $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_{\lambda}$ covering a holomorphic map $f : M_{\lambda_0} \to M_\lambda$ determines a smooth vortex magnetic mode on $\mathcal{I} \subset \mathbb{R}^3_{\lambda_0}$. In terms of the vortex configuration

(Φ, A) *extracted from* $U^{-1}dU$ *using* (5.1.12) *the vortex magnetic mode is*

$$\Psi_H = \Omega^{-1} G \begin{pmatrix} H^*\Phi \\ 0 \end{pmatrix}, \qquad A' = H^* \left(A + \frac{3}{4}\sigma^0 \right). \tag{5.2.35}$$

In the $\lambda_0 = -1$ case this reduces to Corollary 3.8 while for $\lambda_0 = \lambda = 1$ it reduces to Corollary 4.12. The cases $\lambda_0 = 1,\ \lambda = -1$ and $\lambda_0 = 1,\ \lambda = 0$ follow the same logic as the proof of the $\lambda_0 = \lambda = 1$ case but with a different coefficient of the first term in the non-linear equation.

Chapter 6

Gauge fields with linked and knotted field lines

In this Chapter we have a change of tact and turn our attention to magnetic fields whose field lines have non-trivial topology. In particular we encounter magnetic fields where the field lines are linked or there is a field line which is knotted. We say that these magnetic fields are of the Rañada type.

The first Section introduces Rañada's model of linked and knotted electromagnetic fields [3, 4]. This was originally proposed as an attempt to classify electromagnetic fields by their Hopf number, also known as their magnetic helicity, [4]. While these linked and knotted electromagnetic fields do not provide a basis for all solutions of the Maxwell equations [45] they do provide a means of constructing lots of interesting electromagnetic fields. The subject has seen an increase in popularity due to the possibility of linked and knotted electromagnetic fields in magnetohydrodynamics [46, 47].

In the second Section of this Chapter we see how the Popov vortices of Chapter 3 give rise to solutions of a Schrödinger-like equation in the background of a linked magnetic field. In this toy model the spinor wave function picks up the topology of the Popov vortices and vanishes along the vortex lines. This gives a simple example of a system where both the background magnetic field and the wave function have non-trivial topology.

The final Section summarises recent work in [19] where a potential realisation scheme for fields of Rañada type was proposed. Here the magnetic fields arise as synthetic gauge fields in a system consisting of an ultracold atomic gas coupled to two lasers. It is hoped that this realisation scheme can be put in to practice to observe synthetic magnetic fields with linked and knotted structures in a laboratory setting.

6.1 Rañada's knotted light

We start this Chapter with a discussion of Rañada's knotted light, originally developed in [3, 4, 48]. There has been renewed interest starting with [39, 49]. There are two immediate questions: How does this Rañada story relate to the topics of the previous Chapters? And are there physical systems where these linked and knotted configurations show up?

The first question has already been answered in Chapter 3, where we encountered Rañada fields constructed from Popov vortices in the discussion of magnetic vortex modes. This is the direction that we explore in the second Section of this Chapter. There is a sense in which all of the magnetic fields for Popov vortices are of the Rañada type, as they are all constructed from the pullback of the area element ω on S^2. The second question is the topic of the final Section of this chapter. In [19] a realisation scheme was proposed where linked and knotted magnetic fields can arise as synthetic gauge fields in an ultracold atom set up.

The starting point of Rañada's construction is the Maxwell equations on $\mathbb{R}^{1,3}$,

$$dF = 0, \qquad d \star F = \star J. \tag{6.1.1}$$

Rañada's method to solve the source free, $J = 0$, equations is to consider two scalar fields

$$\varphi, \theta : \mathbb{R}^{1,3} \to S^2 \tag{6.1.2}$$

and use these to pull back the volume form[1] on S^2,

$$\omega = \frac{2i dz \wedge d\bar{z}}{(1 + |z|^2)^2}, \tag{6.1.3}$$

[1]With this choice of ω we have that $\int_{S^2} \omega = 4\pi$. Another common choice would be to divide our ω by 4π and get a volume form which integrates to 1.

to $\mathbb{R}^{1,3}$. Let

$$F = \varphi^*\omega \quad \text{and} \quad G = \theta^*\omega, \tag{6.1.4}$$

then impose that $F = \star G$. This gives a non-linear condition on φ and θ which restricts the possible scalar fields, and solves the source free equations by construction, as F is both closed and co-closed.

The vector magnetic field corresponding to F is

$$\vec{B} = 2i\frac{\nabla\varphi \times \nabla\bar{\varphi}}{(1+|\varphi|^2)^2}. \tag{6.1.5}$$

We focus on the static case (restriction to $\mathbb{R}^3$) and only discuss the first part of (6.1.1). In principle there is a current, J, coming from the second part of (6.1.1), but we do not consider it here. This means that we want to solve $dF = 0$ on $\mathbb{R}^3$, using $\varphi : \mathbb{R}^3 \to S^2$. The next step is to identify $\mathbb{R}^3$ with the S^3 of radius ℓ using stereographic projection. As part of this identification we restrict the possible maps φ to those with a well defined fall off as $r \to \infty$. Associated to maps $S^3 \to S^2$ there is a topological invariant the Hopf number, [50]. The Hopf number is

$$h(F) = \frac{1}{16\pi^2}\int_{\mathbb{R}^3} A \wedge F, \qquad \text{where } F = dA. \tag{6.1.6}$$

The prefactor of $\frac{1}{16\pi^2}$ and the fall off of the fields ensure that $h(F) \in \mathbb{Z}$. In [50] they take F to be the pull back of the two-form on S^2 which integrates to 1, $\frac{1}{4\pi}\omega$, so the prefactor is there to account for this difference.

There is a subtlety in evaluating this integral if we take $A = \varphi^*\Gamma$, as it is not globally defined and has singularities. A careful discussion of one way to get around this is given in [51]. We will mention an alternative choice of gauge potential that avoids these singularities when we give some examples.

We encountered magnetic fields of this kind during the discussion of magnetc zero modes on $\mathbb{R}^3$ in Chapter 3. There the composition of the bundle map U and the Hopf projection π, $\pi \circ U$, was said to have Hopf number n^2. The preimage of a given point on the 2-sphere is n circles in S^3 which each link once with the n circles in the preimage of any other point on the two sphere. We shall see this explicitly in some examples below.

For the field strength $F = \varphi^*\omega$, the magnetic field lines are the level curves of φ. To see this note that $\vec{B} \cdot \nabla\varphi = 0$ or equivalently that $F \wedge d\varphi = 0$.

A case of particular interest is when φ is related to a rational map $R : S^2 \to S^2$ through:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[d, "H"] \arrow[drr, bend left, "\varphi"] & & \\ S^3 \arrow[r, "\pi"] & S^2 \arrow[r, "R"] & S^2 \end{tikzcd} \tag{6.1.7}$$

When R is a degree n rational map, the linking number is n^2. Magnetic fields with linked and knotted field lines are some times called Hopfions as they are constructed in the same way as Hopfions in the Skyrme-Faddeev model [18].

At this point it is useful to see some examples.

Example 6.1. Take the scalar field to be

$$\varphi = \pi \circ H : \mathbb{R}^3 \to S^2, \qquad (x, y, z) \mapsto \frac{2\ell\,(x + iy)}{2z\ell + i\,(r^2 - \ell^2)} = \frac{z_2}{z_1}. \tag{6.1.8}$$

The magnetic field is

$$F = \varphi^*\omega = \frac{2i d\varphi \wedge d\bar{\varphi}}{(1 + |\varphi|^2)^2}, \tag{6.1.9}$$

which has Hopf number ,

$$h(F) = 1. \tag{6.1.10}$$

As mentioned above, in the computation of the Hopf number we need to be careful, as the connection $A = -i\frac{(\bar{\varphi}d\varphi - \varphi d\bar{\varphi})}{1+|\varphi|^2}$ is singular on the circle $x^2 + y^2 = \ell^2$. In [51] the authors show how to remove this singularity using a singular gauge transformation. This is essentially the same procedure as was used in Lemma 3.10 when relating a Popov vortex to a vortex configuration. The vortex configuration picture gives an alternative expression for A that does not have a singularity. This global potential is $A = -H^*\sigma^3 = -i\,(\bar{z}_1 dz_1 + \bar{z}_2 dz_2 - z_1 d\bar{z}_1 - z_2 d\bar{z}_2)$ and it agrees with the other choice after the singular gauge transformation has been performed.

Being more explicit about the identification of the complex coordinates z_1, z_2

with x, y, z on $\mathbb{R}^3$, we have that

$$z_1 = \frac{\ell^2 - r^2 + 2i\ell z}{\ell^2 + r^2}, \qquad z_2 = \frac{2i\ell\,(x+iy)}{\ell^2 + r^2}. \tag{6.1.11}$$

The ℓ here is the radius of the three sphere that we are stereographically projecting to $\mathbb{R}^3$. The field lines of the magnetic field in Example 6.1 are the fibres of the Hopf fibration and were plotted in Figure 3.3.

Example 6.2. This time take the scalar field to be

$$\varphi = R \circ \pi \circ H : \mathbb{R}^3 \to S^2, \tag{6.1.12}$$

$$\varphi(x,y,z) = \frac{4\ell^2\,(x+iy)^2}{4\ell^2\,(x+iy)^2 - \left(2z\ell + i\left(r^2 - \ell^2\right)\right)^2} = \frac{z_2^2}{z_1^2 + z_2^2}. \tag{6.1.13}$$

The magnetic field

$$F = \varphi^* \omega \tag{6.1.14}$$

now has Hopf number

$$h(F) = 4. \tag{6.1.15}$$

A plot of the level surface $|\varphi(x,y,z)| = 1.2$ is given in Figure 6.1, the field lines are linked curves which lie on this surface.

Again, to avoid issues coming from the connection having singularities, it is best to work with

$$A = \frac{-i}{|P_1|^2 + |P_2|^2}\left(\bar{P}_1 dP_1 + \bar{P}_2 dP_2 - P_1 d\bar{P}_1 - P_2 d\bar{P}_2\right), \tag{6.1.16}$$

where $P_1 = z_1^2 + z_2^2$ and $P_2 = z_2^2$.

The general story is to take

$$\varphi = R \circ \pi \circ H : \mathbb{R}^3 \to S^2, \qquad (x,y,z) \mapsto R\left(\frac{z_2}{z_1}\right), \tag{6.1.17}$$

with the relationship between the complex coordinates and the stereographic coordinates given in (6.1.11).

Another subtlety of these magnetic fields is that they are zero at the zeros of $\frac{dR}{dz}$.

As was discussed, in Chapters 2 and 3 the zeros of $\frac{dR}{dz}$ correspond to the locations of the Popov vortices that are constructed from R. This can be interpreted as the vortex profile being imposed on a background field profile which is the same as that in Example 6.1.

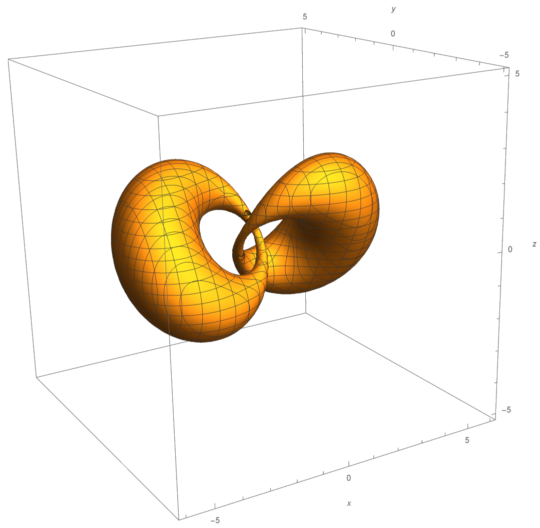

Figure 6.1: A plot of the level surface $|\varphi| = 1.2$ with φ given by (6.1.13) which consists of two linked rings. The field lines of $F = \varphi^*\omega$ are curves on this surface.

To find more magnetic fields with interesting configurations of field lines, we can follow the prescription from [18]. The (a, b) torus knot for a, b co-prime is encoded as $\varphi(z_1, z_2) = \infty$ for

$$\varphi(z_1, z_2) = \frac{z_1^\alpha z_2^\beta}{z_1^b + z_2^a}. \tag{6.1.18}$$

The associated magnetic field $F = \varphi^*\omega$ has Hopf number $h(F) = \alpha a + \beta b$ [18]. If we drop the assumptions that a and b are co-prime we get torus links such as (6.1.13).

For knots the knotted field line is the level curve $\varphi(z_1, z_2) = \infty$ where $z_1^b + z_2^a = 0$. The magnetic field strength is concentrated around this knotted field line [51]. With links, all of the field lines have the linked structure, but the magnetic field strength still tends to concentrate around the pre-image of infinity.

The trefoil knot, $(a, b) = (3, 2)$, is shown in Figure 6.2. Moving from $z_2^3 + z_1^2 = 0$ to $z_2^3 + z_1^2 = 1.5$ the trefoil knot is no longer the level set, it is now a single ring as shown in Figure 6.3.

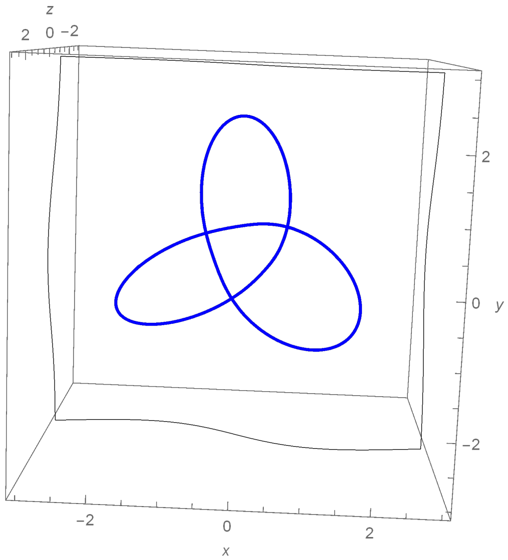

Figure 6.2: A plot of a trefoil knot coming from the level curve $z_1^2 + z_2^3 = 0$.

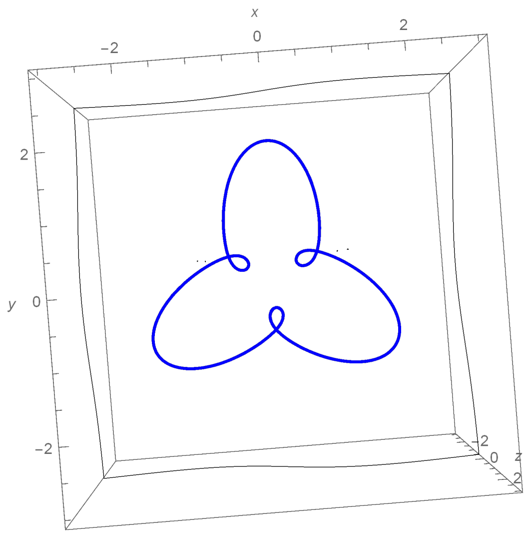

Figure 6.3: A plot of the level set $z_1^2+z_2^3 = 1.5$, where the trefoil knot has disappeared and we now see a single ring.

6.2 A Toy model

In this Section we present a toy model for fields with non-trivial topology in the form of a Schrödinger-like equation for a spinor wave function. This model possesses magnetic fields with linked field lines and exhibits exact solutions. These exact solutions have the topology of a Popov vortex imprinted on the wave function. Popov vortices show up as the Schrödinger-like equation is constructed from the vortex magnetic modes of Chapter 3.

We start by considering a vortex magnetic mode (Ψ, A) on $\mathbb{R}^3$. The pair (Ψ, A) solve the coupled equations

$$\not{D}_A \Psi = 0, \tag{6.2.1}$$

$$\vec{B} = -2i\Psi^\dagger \vec{t}\Psi + \vec{b}. \tag{6.2.2}$$

The vortex magnetic mode is constructed from a vortex configuration, $(\Phi, \tilde{A})$ through Corollary 3.8 as

$$\Psi = \frac{4\ell\left(\ell\mathbb{I} - 2\vec{x}\cdot\vec{t}\right)}{(\ell^2 + r^2)^{\frac{3}{2}}} \begin{pmatrix} H^*\Phi \\ 0 \end{pmatrix}, \tag{6.2.3}$$

$$A = H^*\tilde{A} + \frac{3}{4}H^*\sigma^3. \tag{6.2.4}$$

Using the explicit form of Ψ, the magnetic field becomes

$$\vec{B} = \left(1 - 4|H^*\Phi|^2\right)\vec{b}, \tag{6.2.5}$$

and, as the Maxwell equations are linear, its field lines are the superposition of those of the background field $\vec{b}$ (plotted in Figure 3.3) and a Rañada field coming from the same rational map R as the Popov vortex configuration Φ.

To arrive at the equation for our model, consider

$$\not{D}_A \not{D}_A \Psi = 0 \tag{6.2.6}$$

and use $t_a t_b = \frac{1}{2}\varepsilon_{ab}{}^{c} t_c - \frac{\delta_{ab}}{4}\mathbb{I}$ to arrive at

$$\left(\nabla + i\vec{A}\right)^2 \Psi = 2i\vec{t}\cdot\vec{B}\Psi. \tag{6.2.7}$$

The explicit form of the magnetic field and the spinor are such that

$$2i\vec{t}\cdot\vec{B}\Psi = \frac{1}{\Omega^2}\left(\frac{1}{4} - |H^*\Phi|\right)\Psi. \tag{6.2.8}$$

We thus arrive at

$$-(\vec{\nabla} + i\vec{A})^2\Psi + \frac{1}{\Omega^2}\left(\frac{1}{4} - |H^*\Phi|^2\right)\Psi = 0, \tag{6.2.9}$$

where the spinor wave function Ψ is constructed from a Popov vortex as in (6.2.3). Equation (6.2.9) looks like a zero energy Schrödinger equation with the potential

$$V = \frac{1}{\Omega^2}\left(\frac{1}{4} - |H^*\Phi|^2\right). \tag{6.2.10}$$

Observe that the spinor vanishes at the maximum of the potential, the vortex lines where $H^*\Phi = 0$. This gives a toy model where both the magnetic field and the wave function have non-trivial topology. As the solutions are constructed from a Popov vortex we only encounter magnetic fields whose field lines are links, not knots. This is because the form of a Torus knot in (6.1.18) does not lift to the ratio of homogeneous polynomials on S^3 and thus does not give rise to a vortex configuration. In the next Section we will encounter a very similar looking Schrödinger equation to (6.2.9) where we can have knotted magnetic field lines.

6.2.1 Examples

We now give examples of solutions of (6.2.9) for specific choices of the polynomials P_1, P_2 and discuss the topology that is picked up by the spinorial wave function.

Example 6.3. Start with the simplest case, that of no vortex. Here we can take $H^*\Phi = 1$ and $P_1 = z_1$, $P_2 = z_2$. The magnetic field is thus $\vec{B} = -3\vec{b}$ and its field lines are those plotted in Figure 3.3. In this case the wave function is

$$\Psi = \frac{4\ell}{(\ell^2 + r^2)^{\frac{3}{2}}} \begin{pmatrix} \ell + iz \\ i\,(x + iy) \end{pmatrix}. \tag{6.2.11}$$

The potential is spherically symmetric,

$$V = -\frac{3}{4\Omega^2}, \tag{6.2.12}$$

and is plotted in Figure 6.4.

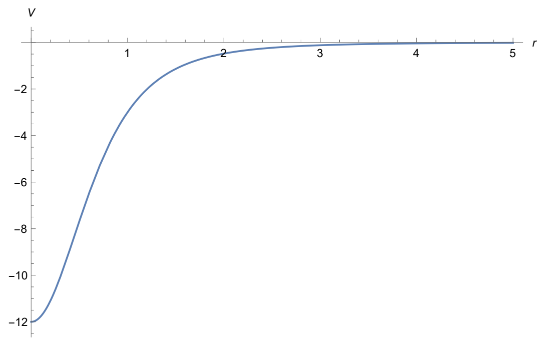

Figure 6.4: A plot of the potential V for the case $H^*\Phi = 1$ with $\ell = 1$.

The wave function in (6.2.11) is spherically symmetric,

$$|\Psi| = \frac{4\ell}{\ell^2 + r^2}, \tag{6.2.13}$$

and the level sets are concentric spheres. In Figure 6.5 we plot one of these level sets.

Example 6.4. For the next example we take the polynomials to be

$$P_2 = z_1 z_2, \qquad P_1 = z_2^2 + z_1 z_2 + z_1^2. \tag{6.2.14}$$

This results in the Higgs field being

$$H^*\Phi = \frac{z_1^2 - z_2^2}{|z_1 z_2|^2 + |z_2^2 + z_1 z_2 + z_1^2|^2}. \tag{6.2.15}$$

The magnetic field is a superposition of the background field, whose field lines we plotted in Figure 3.3, and the Rañada field coming from $\varphi = \frac{H^*P_2}{H^*P_1}$. This second

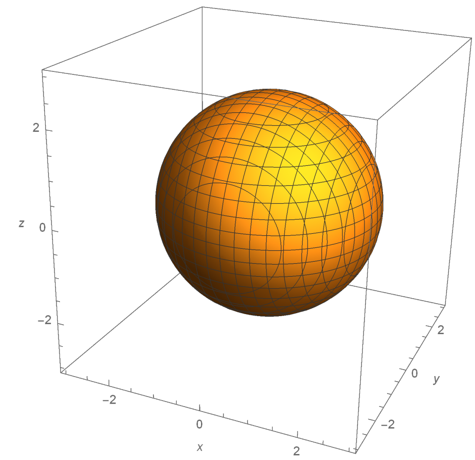

Figure 6.5: A plot of $|\Psi| = \frac{4\ell}{\ell^2+r^2} = 0.1$, where we have set $\ell = 1$.

field has the same topology[2] as that shown in Figure 6.1.

The maximum of the potential is the curve $H^*\Phi = 0$, plotted in Figure 6.6, along which the spinor wave function vanishes.

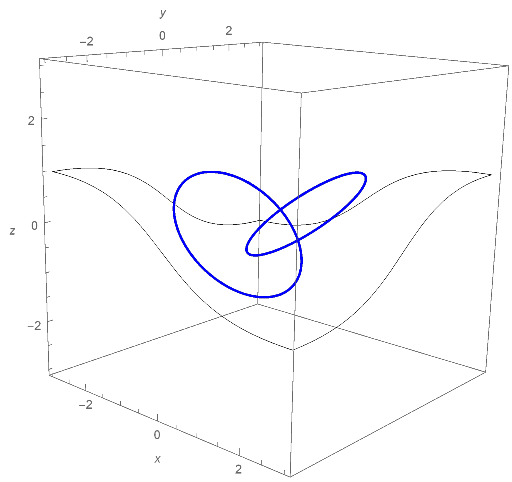

Figure 6.6: A plot of $H^*\Phi = 0$. With $H^*\Phi$ given by (6.2.15). The vortex lines are the linked curves in blue.

6.3 Linked and knotted fields in the Λ-scheme

This Section is based on the paper [19] where a potential realisation scheme of synthetic magnetic fields with linked and knotted field lines was proposed. Throughout this Section we include factors of $\hbar$ in the Schrödinger equation, taking $\hbar = 1$ to

[2] We could have taken the same polynomials as in Example 6.2 but then $H^*\Phi = 0$ is zero on the circle of radius ℓ and the z-axis.

compare with the previous Section. The proposed realisation uses a three level system called the Λ-scheme [20, 52], depicted in Figure 6.7. This involves an ensemble of atoms with three internal energy levels, two almost degenerate ground states $|g_1\rangle, |g_2\rangle$ coupled by lasers to a third excited state $|e\rangle$. The atom-light coupling is controlled by space-dependent, complex Rabi frequencies κ_1, κ_2. Here we assume that the lasers are resonant with the transitions, which results in the atom-light coupling being

$$H_{\text{int}} = \begin{pmatrix} 0 & 0 & \kappa_1 \\ 0 & 0 & \kappa_2 \\ \bar{\kappa}_1 & \bar{\kappa}_2 & 0 \end{pmatrix}. \tag{6.3.1}$$

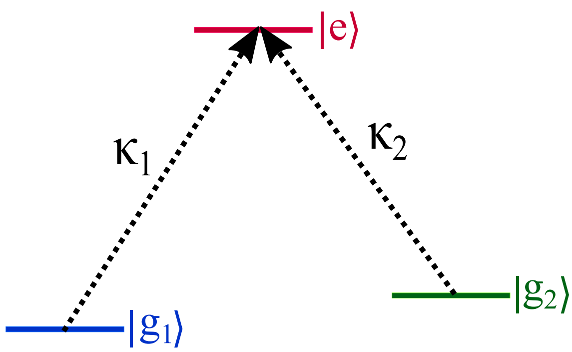

Figure 6.7: A schematic of the Λ-scheme. The $|g_i\rangle$ are the two almost-degenerate ground states, $|e\rangle$ is the excited state and the κ_i are the complex Rabi frequencies of the lasers tuned to the transition between $|g_i\rangle$ and $|e\rangle$. Figure produced by C. Duncan.

The proposal is to realise this system in an ultracold atomic gas, a Bose-Einstein condensate. Following [20], the condensate wave function is written in terms of the eigenstates of H_{int} as

$$|\Psi\rangle = \psi_D|D\rangle + \psi_+|+\rangle + \psi_-|-\rangle. \tag{6.3.2}$$

The eigenstates of H_{int} are

$$|D\rangle = \frac{1}{\sqrt{|\kappa_1|^2 + |\kappa_2|^2}} \begin{pmatrix} \bar{\kappa}_2 \\ -\bar{\kappa}_1 \\ 0 \end{pmatrix}, \tag{6.3.3}$$

$$|\pm\rangle = \frac{1}{\sqrt{2}\sqrt{|\kappa_1|^2 + |\kappa_2|^2}} \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \pm\sqrt{|\kappa_1|^2 + |\kappa_2|^2} \end{pmatrix}, \tag{6.3.4}$$

with eigenvalues

$$\lambda_D = 0, \qquad \lambda_\pm = \pm\sqrt{|\kappa_1|^2 - |\kappa_2|^2}. \tag{6.3.5}$$

The state $|D\rangle$ with eigenvalue zero is referred to as the dark state.

The Schrödinger equation for $|\Psi\rangle$ is

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \left(\frac{\vec{P}^2}{2m} + H_{\text{int}} + V\right)|\Psi\rangle, \tag{6.3.6}$$

where m is the mass of an atom, V is the trapping potential used to keep the condensate together and $\vec{P} = -i\hbar\nabla$ is the momentum operator. Working in the adiabatic approximation, that is we assume that all perturbations act "slowly enough" that we do not leave the eigenstate that we start in, and preparing the condensate in the dark state, we have that [20]

$$|\Psi\rangle \simeq \psi_D|D\rangle. \tag{6.3.7}$$

The Schrödinger equation thus becomes

$$i\hbar\frac{\partial\psi_D}{\partial t}|D\rangle = \frac{\vec{P}^2}{2m}\left(\psi_D|D\rangle\right) + V\psi_D|D\rangle. \tag{6.3.8}$$

Projecting this onto the dark state by taking the inner product with $\langle D|$, we find that

$$i\hbar\frac{\partial\psi_D}{\partial t} = \left(\frac{\left(\vec{P} - \vec{\mathcal{A}}\right)^2}{2m} + \Phi + V\right)\psi_D \tag{6.3.9}$$

with $\vec{\mathcal{A}}$ a Berry connection and Φ the geometric potential given by

$$\vec{\mathcal{A}} = -\langle D|\vec{P}D\rangle = \frac{i\hbar\left(\kappa_1\nabla\bar{\kappa}_1 + \kappa_2\nabla\bar{\kappa}_2 - \bar{\kappa}_1\nabla\kappa_1 - \bar{\kappa}_2\nabla\kappa_2\right)}{2\left(|\kappa_1|^2 + |\kappa_2|^2\right)}, \tag{6.3.10}$$

$$\Phi = -\frac{1}{2m}\left(\vec{\mathcal{A}}^2 + \||\vec{P}D\rangle\|^2\right) = \frac{\hbar^2}{2m}\frac{\nabla\bar{\zeta}\cdot\nabla\zeta}{\left(1+|\zeta|^2\right)^2}, \tag{6.3.11}$$

where $\zeta = \frac{\kappa_1}{\kappa_2}$. We could express $\vec{\mathcal{A}}$ in terms of ζ, however, this would lead to $\vec{\mathcal{A}}$ being in a singular gauge. The form of the gauge potential (6.3.10) leads to the magnetic field being

$$\vec{B} = i\hbar\frac{\nabla\zeta\times\nabla\bar{\zeta}}{\left(1+|\zeta|^2\right)^2}, \tag{6.3.12}$$

which is of the Rañada type[3] (compare with Equation (6.1.5)) with ζ playing the role of $\varphi : \mathbb{R}^3 \to S^2$. This suggests that if we can represent our homogeneous complex polynomials P_1, P_2 as Rabi frequencies, then Rañada fields can be realised in the Λ-scheme. These emergent magnetic fields are known as synthetic magnetic fields.

The Rabi frequencies for physical lasers need to solve the Helmholtz equation or its paraxial approximation, see Appendix C. Unfortunately, our complex polynomials do not satisfy the paraxial Helmholtz equation. To get around this we follow the approach of [21–23], where linked and knotted optical vortex lines were realised in laser beams as a superposition of Laguerre-Gaussian (LG) modes. Our method of constructing the topological synthetic magnetic fields consists of the following steps:

1. Starting from a map $\varphi(\vec{x}) = \frac{P_2(H(\vec{x}))}{P_1(H(\vec{x}))}$, restrict it to the $z = 0$ plane, where z is the direction of propagation for the lasers, the result is a ratio of polynomials in x and y.
2. Expand the polynomials P_2 and P_1 in terms of LG modes restricted to the $z = 0$ plane and without the common Gaussian factor.
3. Replace P_2 and P_1 in φ by the expansions in the LG modes, including their z-dependence (and note that the common Gaussian factor cancels).
4. Check numerically if the level curves of the resulting function ζ have the same topology as the level curves of φ.
5. If they do, realise the level curves as synthetic magnetic field lines through

[3] In [19] units where $\hbar = \frac{1}{2\pi}$ were picked and thus the magnetic field is constructed from the pullback of $\frac{1}{4\pi}\omega$.

$\zeta = \kappa_1/\kappa_2$, where κ_1 and κ_2 are the LG modes approximating P_2 and P_1.

Below we present three examples where this approach works. However, at the moment there is no known proof as to whether this approximation scheme works in general. There has been some work in [53, 54] where it was proved that a solution of the Helmholtz equation, u, can be found such that $u^{-1}(0)$ is diffeomorphic to a curve containing any desired knot. The proof of this is not constructive and we cannot identify which solution we need to consider for a given knot or link. If these solutions satisfy the paraxial approximation then finding them would solve the problem of finding the desired Rabi frequencies.

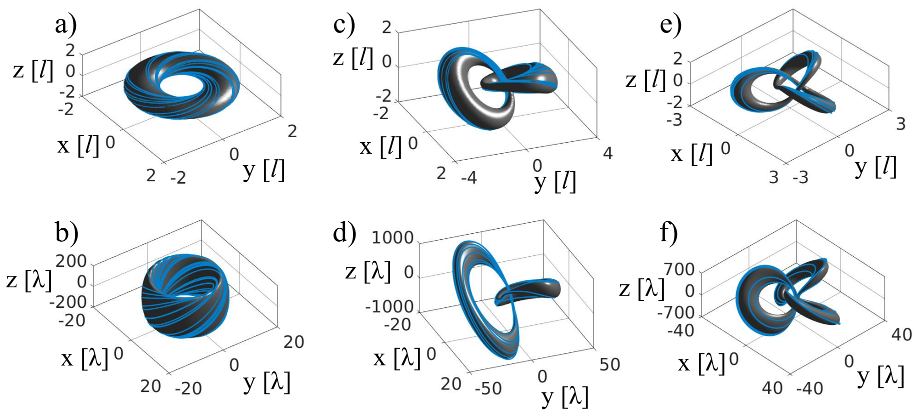

Figure 6.8: Exact and approximated magnetic field lines, realised as level curves of the complex field φ and its Laguerre-Gauss approximation ζ . We show level surfaces of $|\varphi|$ and $|\zeta|$, and, on each level surface, we show magnetic field lines. a) Exact Hopf circles ($\varphi_{\mathrm{H}} = \frac{z_2}{z_1}$). b) Realised Hopf circles (ζ_{H}). c) Exact linked rings ($\varphi_{\mathrm{L}} = \frac{z_2^2}{z_2^2+z_1^2}$). d) Realised linked rings (ζ_{L}). e) Exact trefoil knot ($\varphi_{\mathrm{T}} = \frac{z_2^3}{z_2^3+z_1^2}$). f) Realised trefoil knot (ζ_{T}). The unit of length for the exact magnetic fields is ℓ and for the realised fields the wavelength λ with $\alpha = 100$. Figure produced by C. Duncan.

Carrying out step 2 above we find the following approximations in terms of Laguerre-Gaussians for the Hopf map $\varphi_{\mathrm{H}} = \frac{z_2}{z_1}$, the quadratic Hopf map $\varphi_{\mathrm{L}} = \frac{z_2^2}{z_2^2+z_1^2}$

and the map $\varphi_{\mathrm{T}} = \frac{z_2^3}{z_2^3+z_1^2}$ for trefoil knot:

$$\zeta_{\mathrm{H}} = \frac{2\alpha\mathcal{L}_{0-1}}{i\left[\left(\frac{\alpha^2}{2}-1\right)\mathcal{L}_{00} - \frac{\alpha^2}{2}\mathcal{L}_{10}\right]}, \tag{6.3.13}$$

$$\zeta_{\mathrm{L}} = \frac{(2\alpha)^2\mathcal{L}_{0-2}}{(2\alpha)^2\mathcal{L}_{0-2} + c_0\mathcal{L}_{00} + c_1\mathcal{L}_{10} + c_2\mathcal{L}_{20}}, \tag{6.3.14}$$

$$\zeta_{\mathrm{T}} = \frac{(2\alpha)^3\mathcal{L}_{0-3}}{(2\alpha)^3\mathcal{L}_{0-3} + c'_0\mathcal{L}_{00} + c'_1\mathcal{L}_{10} + c'_2\mathcal{L}_{20} + c'_3\mathcal{L}_{30}}, \tag{6.3.15}$$

where we have defined $\alpha = \omega_0/\ell$ with ω_0 the beam waist of the laser and ℓ the length scale introduced through our stereographic coordinates as the radius of the three sphere. The coefficients c_i and c'_i are polynomials in α;

$$c_0 = -\frac{\alpha^4}{2} + \alpha^2 - 1, \tag{6.3.16}$$

$$c_1 = \alpha_2 - \alpha_4, \tag{6.3.17}$$

$$c_2 = \frac{\alpha^4}{2}, \tag{6.3.18}$$

$$c'_0 = \frac{1}{4}\left(-4 + 2\alpha^2 + 2\alpha^4 - 3\alpha^6\right), \tag{6.3.19}$$

$$c'_1 = \alpha^2\left(-2 - 4\alpha^2 + 9\alpha^6\right), \tag{6.3.20}$$

$$c'_2 = 2\alpha^4 - 9\alpha^6, \tag{6.3.21}$$

$$c'_3 = 3\alpha^6. \tag{6.3.22}$$

A comparison of the exact and approximated magnetic fields for all three cases considered is shown in Figure 6.8. For all approximated fields we have chosen a beam width of $\alpha = 100$ for the LG modes and work in units of wavelength of the laser λ. The approximated fields are stretched out in the z-direction compared to the exact fields. For all three examples considered, the topological nature of the realised magnetic field lines is clear, as the level set of each have similar forms to those of the exact fields.

An unexpected consequence of the topology of the synthetic gauge field is that the ground-state condensate wave function appears to pick up the same topology by vanishing at the maxima of the scalar potential Φ, which coincides with the level curve $\zeta^{-1}(\infty)$. Plots of the condensate wave function showing this were included in [19] and movies of the condensate wave function are included in the supplementary material to [19] as numerical evidence of the topology picked up by the wave function.

It is interesting to compare the Schrödinger equation (6.3.9) and the Schrödinger-like equation (6.2.9) in the toy model of the previous Section. To facilitate the comparison, we take units where $\hbar = 1 = m$ and contrast the gauge potentials

$$\vec{\mathcal{A}} = \frac{i\left(\kappa_1 \nabla \bar{\kappa}_1 + \kappa_2 \nabla \bar{\kappa}_2 - \bar{\kappa}_1 \nabla \kappa_1 - \bar{\kappa}_2 \nabla \kappa_2\right)}{2\left(|\kappa_1|^2 + |\kappa_2|^2\right)}, \tag{6.3.23}$$

$$\begin{aligned}\vec{A} =& \frac{i}{4}\left(z_1 \nabla \bar{z}_1 + z_2 \nabla \bar{z}_2 - \bar{z}_1 \nabla z_1 - \bar{z}_2 \nabla z_2\right) \\ & - \frac{2i\left(P_1 \nabla \bar{P}_1 + P_2 \nabla \bar{P}_2 - \bar{P}_1 \nabla P_1 - \bar{P}_2 \nabla P_2\right)}{|P_1|^2 + |P_2|^2}\end{aligned} \tag{6.3.24}$$

Constructing magnetic fields from the same polynomials, κ_1 is an approximation of P_2 and κ_2 is an approximation of P_1 in terms of Laguerre-Gaussian functions. Both potentials give rise to magnetic fields with non-trivial topology. However, the topology of the resulting magnetic fields is different. The toy model has a background field present, whose magnetic field lines are those in Figure 3.3, as well as the field generated by the polynomials, whereas the Λ-scheme gauge potential is just that of the linked field constructed from the polynomials. A key difference is that in (6.3.23) the κ's can be chosen so that $\nabla \times \vec{\mathcal{A}}$ has a knotted field line. While in (6.3.24) this is not allowed because a polynomial P_1, such that $P_1 = 0$ encodes a knot, is not homogeneous in z_1, z_2. As the polynomial is not homogeneous it does not correspond to one of the polynomials defining a Popov vortex.

Another similarity is that in both equations the wave function picks up non-trivial topology. Though the specifics of the topology is again a difference between the two models. In the realisation scheme numerical evidence, [19], suggests that ψ_D inherits the topology of $P_1 = 0$, a specific magnetic field line. In contrast, in the toy model the spinor wave function inherits the topology of the vortex lines.

Chapter 7

Conclusions

In this thesis we have pointed out and developed the relationship between integrable Abelian vortex equations and flat non-Abelian connections, which encode the geometric structure of the manifolds the vortices are on. The first example of this was in Chapter 2 where Theorem 2.3 presented the link between the vortex equations and flat Cartan connections encoding the geometry of the deformed frame defined by the vortex, ϕe_{λ_0}. We then presented a three dimensional generalisation of this story for vortex configurations on group manifolds. This lift leads to manifestly smooth expressions for the connection and the Higgs field even when the vortex in two dimensions has singularities. The three dimensional picture leads to a construction of magnetic modes from the data of a vortex. In the Euclidean case this leads to a geometric understanding of some of the Dirac zero-modes constructed in [13, 32].

A future direction of work suggested by this result is to consider the metric on the moduli space of flat $\mathbb{H}^1_\lambda$ connections on M_{λ_0}, naturally given in terms of the Killing form on the Lie algebra and check if this gives an alternative construction of a metric on the moduli space of vortices. Another direction is to extend the construction of vortex magnetic modes to include the case of Jackiw-Pi vortices. Here this was not done as the metric on $\mathbb{H}^1_0$ is singular. However, centrally extending $\mathbb{H}^1_0$ to the Nappi-Witten space, [55], we have a group manifold with a Lorentzian metric. This central extension can be carried out for all of the groups and is a trivial extension for $SU(2)$ and $SU(1,1)$. Taking the Dirac operator on the Nappi-Witten space it needs to be checked if Jackiw-Pi vortex configurations give rise to magnetic-modes. The complication here is that the spin connection is no longer given by the Maurer-

Vortex type	Instanton gauge group	Cartan group
Hyperbolic	$SU(2)$	$SU(1,1)$
Popov	$SU(1,1)$	$SU(2)$
Jackiw-Pi	$SU(1,1)$	$SU(2)$
Ambjorn-Olsen	$SU(1,1)$	$SU(2)$
Bradlow	SE_2	SE_2

Table 7.1: This table summarises the instanton gauge group from [24] and the group for the flat connection constructed here.

Cartan one form and its construction requires more care.

Before concluding there is time to fulfil a promise from Chapter 2, as well as the conclusions of [10, 15], and give a comparison of the instanton picture of the integrable vortex equations in [24] with the flat connection picture presented here. In Chapter 2 we discussed that in [24] it was shown that all five of the integrable vortex equations can be constructed as the dimensional reduction of an appropriate (anti-) self dual Yang-Mills theory on $M_{\lambda_0} \times M_{-\lambda_0}$. This construction is based on the general story of gauge fields which posses space time symmetries introduced in [56]. In [10, 15] the papers that Chapters 3 and 4 are adapted from it was observed that there is an interesting relationship between the gauge group of the Yang-Mills theory and the Lie algebra that the Cartan connection is valued in. In Table 7.1 we give the five integrable vortex equations, the gauge groups for the instantons from [24] and the group that our flat connection is valued in the Lie algebra of.

While this relationship still seems mysterious and is yet to be fully understood we can spell out some of the details.

In the conventions of this thesis the construction from [24] starts by considering instantons on $\mathbb{R}^4 \simeq M_{\lambda_0} \times M_{-\lambda_0}$ with gauge group $\mathbb{H}^1_{-\lambda}$ that are equivariant with respect to $\mathbb{H}^1_{-\lambda_0}$. This amounts to the instanton being independent of the $M_{-\lambda_0}$ factor and thus reducing to (λ_0, λ) vortices on M_{λ_0}. Explicitly the instanton, [24], is given by the $\mathbb{H}^1_{-\lambda}$-connection

$$\mathcal{A}_{\text{CD}} = -\left(a - \Gamma_{-\lambda_0}\right) t_0 + \frac{i\phi}{2}\bar{e}_{-\lambda_0} t_- - \frac{i\bar{\phi}}{2} e_{-\lambda_0} t_+, \tag{7.0.1}$$

with (ϕ, a) the (λ_0, λ) vortex on M_{λ_0} and $e_{-\lambda_0}, \Gamma_{-\lambda_0}$ the complexified frame and spin connection on $M_{-\lambda_0}$. This is not the only connection which can be constructed out of the data of a vortex. In Theorem 2.3 we saw that a (λ_0, λ) vortex is equivalent to a flat $\mathbb{H}^1_\lambda$-connection

$$A = -\left(a + \Gamma_{\lambda_0}\right) t_0 + \frac{i\phi}{2} e_{\lambda_0} t_- - \frac{i\bar{\phi}}{2} \bar{e}_{\lambda_0} t_+. \tag{7.0.2}$$

It is interesting to contrast the two connections. $\mathcal{A}_{\mathrm{CD}}$ is an anti- self dual connection on a conformally flat four-manifold while A is a flat connection on a Riemann surface. This manifests itself in the fact that A only depends on information on M_{λ_0}, the vortex (ϕ, a) and the frame field and spin connection. On the other hand $\mathcal{A}_{\mathrm{CD}}$ depends on information from both M_{λ_0} and $M_{-\lambda_0}$. Another difference that should be noted is that while we have used the same notation for the generators, $t_0, t_\pm$ they are not exactly the same, the key difference is in t_+. For $\mathbb{H}^1_\lambda$ the explicit form of t_+ is

$$t_+ = \begin{pmatrix} 0 & i\lambda \\ 0 & 0 \end{pmatrix}. \tag{7.0.3}$$

This means the sign in t_+ is different for the two connections.

From Chapters 3, 4 and 5 we know that the flat connection in two dimensions is related to a flat connection on the group manifold $\mathbb{H}^1_{\lambda_0}$ given by (5.1.12). An immediate question of interest is if there is a way to go directly between the instanton and the three dimensional Cartan connection. At the moment we only know how to pass between them by going through the vortex in two dimensions. There are definitely key differences in their construction with A being constructed from the pullback of the left-invariant Maurer-Cartan one-form on $\mathbb{H}^1_\lambda$ while $\mathcal{A}_{\mathrm{CD}}$, following the general construction in [56], is constructed from left-invariant data from $\mathbb{H}^1_{-\lambda_0}$ and right-invariant data from $\mathbb{H}^1_{-\lambda}$.

Finally consider the diagram

$$\begin{array}{ccc} \mathbb{H}^1_{-\lambda_0} \times \mathbb{H}^1_{\lambda_0} & \xrightarrow{U} & \mathbb{H}^1_\lambda \times \mathbb{H}^1_{-\lambda} \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ M_{-\lambda_0} \times M_{\lambda_0} & \xrightarrow{f} & M_\lambda \times M_{-\lambda} \end{array} \tag{7.0.4}$$

where $f : M_{\lambda_0} \to M_\lambda$ is the vortex and $U : \mathbb{H}^1_{\lambda_0} \to \mathbb{H}^1_\lambda$ is the bundle map that we encountered in Theorem 5.4. Now from the instanton point of view $\mathbb{H}^1_{-\lambda_0}$ would be the symmetry group and $\mathbb{H}^1_{-\lambda}$ is the gauge group. This could be flipped round to

$$\begin{tikzcd} \mathbb{H}^1_{\lambda_0} \times \mathbb{H}^1_{-\lambda_0} \arrow[d, "\pi"] & \mathbb{H}^1_{-\lambda} \times \mathbb{H}^1_\lambda \arrow[l, "V"'] \arrow[d, "\pi"] \\ M_{\lambda_0} \times M_{-\lambda_0} & M_{-\lambda} \times M_\lambda \arrow[l, "g"'] \end{tikzcd} \tag{7.0.5}$$

with $g : M_{-\lambda} \to M_{-\lambda_0}$ a vortex and $V : \mathbb{H}^1_{-\lambda} \to \mathbb{H}^1_{-\lambda_0}$ a bundle map. Where now the instanton point of view has $\mathbb{H}^1_\lambda$ as the symmetry group and $\mathbb{H}^1_{\lambda_0}$ as the gauge group. This suggests that at the level of the groups there is an interesting relationship between the different vortex equations.

Appendix A

Additional Lorentzian results

In this Appendix we collect some results related to Lorentzian vortex configurations which were not included in either Chapter 4 or the paper [15] that the Chapter was based on. The conventions of this Appendix are the same as those of Chapter 4.

A.1 Equivariant functions

First up we present a Lemma which justifies the commutative diagram relating equivariant functions on AdS_3 and smooth functions on H^2.

Lemma A.1

Consider the equivariant function over AdS_3

$$C^{\infty}(AdS_3, \mathbb{C})_N = \{F : AdS_3 \to \mathbb{C} | 2i\mathfrak{X}_0 F = -NF\}, \quad N \in \mathbb{N}^0. \tag{A.1.1}$$

Using the section

$$s : H^2 \to SU(1,1), \quad z \mapsto \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix}, \tag{A.1.2}$$

of the Lorentzian Hopf fibration, $\pi : SU(1,1) \to H^2, (z_1, z_2) \mapsto \frac{z_2}{z_1}$ we can establish the relationship

$$s^* \left(\mathfrak{X}_+ F\right)(z) = i \left((1-|z|^2)\bar{\partial} - \frac{N}{2} z \right) s^* \left(F\right)(z). \tag{A.1.3}$$

From now on we use $q = (1 - |z|^2)$ to shorten the expressions.

Proof. To start note that we frequently interchange the z_i with their pullback via s as that is all we need to consider in this case. First consider that since $F \in C^\infty(AdS_3, \mathbb{C})_N$ we have that

$$i\mathfrak{X}_0 F = -\frac{N}{2} F, \tag{A.1.4}$$

this allows us to write

$$i\left(q\bar{\partial} - \frac{N}{2}z\right) s^*\left(F\right)(z) = i\left(q\bar{\partial}s^*F + izs^*(\mathfrak{X}_0 F)\right). \tag{A.1.5}$$

Next consider that

$$\bar{\partial} = \bar{\partial}z_1\partial_1 + \bar{\partial}z_2\partial_2 + \bar{\partial}\bar{z}_1\bar{\partial}_1 + \bar{\partial}\bar{z}_2\bar{\partial}_2, \tag{A.1.6}$$

and compute the coefficients

$$\bar{\partial}z_1 = \bar{\partial}\frac{1}{\sqrt{q}} = \frac{1}{2}\frac{z}{q^{\frac{3}{2}}}, \tag{A.1.7}$$

$$\bar{\partial}z_2 = \frac{1}{2}\frac{z^2}{q^{\frac{3}{2}}}, \tag{A.1.8}$$

$$\bar{\partial}\bar{z}_1 = \frac{1}{2}\frac{z}{q^{\frac{3}{2}}}, \tag{A.1.9}$$

$$\bar{\partial}\bar{z}_2 = \bar{\partial}\frac{\bar{z}}{\sqrt{q}} = \frac{1}{q^{\frac{3}{2}}} - \frac{|z|^2}{2q^{\frac{3}{2}}}, \tag{A.1.10}$$

to arrive at

$$q\bar{\partial}s^*F = \frac{1}{2\sqrt{q}}\left(z\partial_1 + zz\partial_2 + z\bar{\partial}_1 + 2\bar{\partial}_2 - |z|^2\bar{\partial}_2\right)s^*F. \tag{A.1.11}$$

We then expand

$$izs^*(\mathfrak{X}_0 F) = -\frac{1}{2\sqrt{q}}\left(zz\partial_2 + z\partial_1 - |z|^2\bar{\partial}_2 - z\bar{\partial}_1\right)s^*F, \tag{A.1.12}$$

before constructing the full term as

$$i\left(q\bar{\partial}s^*F + izs^*(\mathfrak{X}_0F)\right) = \frac{i}{2\sqrt{q}}\left(z\partial_1 + zz\partial_2 + z\bar{\partial}_1 + 2\bar{\partial}_2 - |z|^2\bar{\partial}_2\right.$$

$$\left. - zz\partial_2 - z\partial_1 + |z|^2\bar{\partial}_2 + z\bar{\partial}_1\right) s^*F, \tag{A.1.13}$$

$$= \frac{i}{\sqrt{q}}\left(z\bar{\partial}_1 + \bar{\partial}_2\right) s^*F, \tag{A.1.14}$$

$$= i\left(z_2\bar{\partial}_1 + z_1\bar{\partial}_2\right) s^*F, \tag{A.1.15}$$

$$= s^*(\mathfrak{X}_+F)(z). \tag{A.1.16}$$

This gives the desired result. □

A.2 Decomposition of the gauge potential

Here we present the analogue of Lemma 3.5 from Chapter 3 along with its proof.

Lemma A.2

Take (Φ, A) to be a Lorentzian vortex configuration of degree $N-1$ on $SU(1,1)$. Consider the modulus-argument decomposition of the Higgs field, Φ, which is valid away from the zeros of Φ,

$$\Phi = e^{\frac{M}{2}+i\chi}. \tag{A.2.1}$$

Then the vortex gauge potential A can be expressed using the formula

$$A = \frac{\ell}{4}\star\left(\varsigma^0 \wedge dM\right) + d\chi, \tag{A.2.2}$$

which is valid away from the zeros of Φ.

Proof. First off we note that from Corollary 4.4 in Chapter 4 we know that the gauge potential can be expressed as

$$A = (N-1)\varsigma^0 - \frac{i}{2}\mathfrak{X}_-\ln D^2\varsigma + \frac{i}{2}\mathfrak{X}_+\ln D^2\bar{\varsigma}, \tag{A.2.3}$$

and the Higgs field can be expressed as

$$\Phi = \frac{F_1 \partial_2 F_2 - F_2 \partial_2 F_1}{z^1 D^2}, \tag{A.2.4}$$

where

$$D^2 = |F_1|^2 - |F_2|^2, \tag{A.2.5}$$

and the F_i are equivariant holomorphic functions of degree N on $SU(1,1)$. Now note that since the F_i are holomorphic

$$\mathfrak{X}_- \ln(D^2) = -\mathfrak{X}_- \ln \bar{\Phi}, \quad \mathfrak{X}_+ \ln(D^2) = -\mathfrak{X}_+ \ln \Phi. \tag{A.2.6}$$

This means that we can rewrite the gauge potential as

$$A = (N-1)\varsigma^0 + \frac{i}{2}\mathfrak{X}_- \ln \bar{\Phi} \varsigma - \frac{i}{2}\mathfrak{X}_+ \ln \Phi \bar{\varsigma}. \tag{A.2.7}$$

If we use the parametrisation of (A.2.1) then we get that

$$A = (N-1)\varsigma^0 + \left(\frac{i}{4}\mathfrak{X}_- M + \frac{1}{2}\mathfrak{X}_- \chi\right)\varsigma - \left(\frac{i}{4}\mathfrak{X}_+ M - \frac{1}{2}\mathfrak{X}_+ \chi\right)\bar{\varsigma}, \tag{A.2.8}$$

$$= (N-1)\varsigma^0 + \frac{i}{4}\left(\mathfrak{X}_- M\varsigma - \mathfrak{X}_+ M\bar{\varsigma}\right) + \frac{1}{2}\left(\mathfrak{X}_-\chi + \mathfrak{X}_+\varsigma\chi\bar{\varsigma}\right). \tag{A.2.9}$$

Taking the Hodge-$\star$ relative to the orientation

$$\frac{\ell^3}{8}\varsigma^0 \wedge \varsigma^1 \wedge \varsigma^2, \tag{A.2.10}$$

it can be shown that

$$\star(\varsigma^0 \wedge \varsigma) = \frac{2i}{\ell}\varsigma, \quad \star(\varsigma^0 \wedge \bar{\varsigma}) = -\frac{2i}{\ell}\bar{\varsigma}. \tag{A.2.11}$$

Using this and the fact that for any differentiable function $f : SU(1,1) \to \mathbb{C}$,

$$df = X_0 f \varsigma^0 + \frac{1}{2}X_- f \varsigma + \frac{1}{2}X_+ f \bar{\varsigma}, \tag{A.2.12}$$

we get that

$$\frac{i}{4}\left(X_{-}M\varsigma - X_{+}M\bar{\varsigma}\right) = \frac{\ell}{4}\star\left(\varsigma^{0}\wedge dM\right). \tag{A.2.13}$$

For χ we use the equivariance of Φ, and the fact that $X_0 M = 0$ to see that

$$1 - N = iX_0 \ln \Phi = -X_0\chi, \tag{A.2.14}$$

combining this with (A.2.12) we have that

$$(N-1)\varsigma^{0} + \frac{1}{2}\left(X_{-}\chi + X_{+}\varsigma\chi\bar{\varsigma}\right) = d\chi. \tag{A.2.15}$$

Putting Equations (A.2.9), (A.2.13) and (A.2.15) together we arrive at the claimed expression that

$$A = \frac{\ell}{4}\star\left(\varsigma^{0}\wedge dM\right) + d\chi. \tag{A.2.16}$$

□

It is interesting to note that the main difference between this result and the Euclidean result is an overall sign difference in A.

A.3 Lorentzian Loss and Yau result

An analogue of the trick introduced by Loss and Yau [13] can now be employed to obtain zero-modes of a coupled Dirac operator from eigenmodes of the ordinary Dirac operator. Begin by setting

$$\frac{4}{\ell}iA_i = \rho\frac{\Psi^{\dagger}\mathfrak{t}_i\Psi}{\Psi^{\dagger}\Psi}, \tag{A.3.1}$$

where Ψ is not null, the identity $\Psi^{\dagger}\mathfrak{t}^{i}\Psi\mathfrak{t}_i = -2\left(\Psi\Psi^{\dagger} - \frac{1}{2}\Psi^{\dagger}\Psi\mathbb{I}\right)$ can be used to arrive at

$$\frac{4}{\ell}i\vec{A}\cdot\vec{\mathfrak{t}}\Psi = -\rho\Psi, \tag{A.3.2}$$

which implies that

$$\not{D}_{SU(1,1)}\Psi = \rho\Psi \Leftrightarrow \not{D}_{SU(1,1),A}\Psi = 0. \tag{A.3.3}$$

An explicit formula on $\mathbb{R}^{1,2}$ for the gauge potential in terms of the spinor analogous to the result in [13] is given in Equation (4.5.13).

Lemma A.3

A non-null spinor $\Psi \in \mathbb{C}^{1,1}$ with a divergence free spin density, $\Sigma_i = 2i\Psi^\dagger \mathfrak{t}_i \Psi$ with $\partial^i \Sigma_i = 0$, will solve a coupled Dirac equation with potential

$$A_i = \frac{1}{|\vec{\Sigma}|}\big(\frac{1}{2}\varepsilon_i^{\ jk}\partial_j \Sigma_k + \mathrm{Im}(\Psi^\dagger \partial_i \Psi)\big). \tag{A.3.4}$$

The proof of this is very similar to the one presented in [57] for the Euclidean case.

Proof. We start from the observation that if we have

$$\Sigma_i = 2i\Psi^\dagger \mathfrak{t}_i \Psi, \tag{A.3.5}$$

$$\partial^i \Sigma_i = 0, \tag{A.3.6}$$

then it can be verified that

$$\Psi = \frac{1}{\sqrt{2(|\vec{\Sigma}| + \Sigma_0)}}\begin{pmatrix} |\vec{\Sigma}| + \Sigma_0 \\ \Sigma_1 + i\Sigma_2 \end{pmatrix}, \tag{A.3.7}$$

gives an expression for the spinor in terms of the spin density. From the above expression for Ψ it follows that

$$|\Psi|^2 = -|\vec{\Sigma}|, \tag{A.3.8}$$

and that

$$\Psi \otimes \Psi^\dagger = -\frac{1}{2}|\vec{\Sigma}|\mathbb{I} + i\Sigma_i \mathfrak{t}^i. \tag{A.3.9}$$

The algebra of $su(1,1)$ generators leads to

$$\mathfrak{t}_i \mathfrak{t}_j = \frac{1}{2}\varepsilon_{ij}^{\ \ k}\mathfrak{t}_k - \frac{1}{4}\eta_{ij}\mathbb{I}, \tag{A.3.10}$$

which allows the divergence free condition to be grouped together with an expression for the curl of Σ as

$$\mathbf{t}^i\mathbf{t}^j\partial_i\Sigma_j = \frac{1}{2}J_k\mathbf{t}^k, \tag{A.3.11}$$

where $J_i = \varepsilon_i{}^{jk}\partial_j\Sigma_k$. Next we can rearrange the tensor product equation to get

$$i\Sigma_i\mathbf{t}^i = \Psi\otimes\Psi^\dagger + \frac{1}{2}|\vec{\Sigma}|\mathbb{I}, \tag{A.3.12}$$

substituting this into equation (A.3.11) results in

$$\mathbf{t}^i\big(\partial_i(\Psi\otimes\Psi^\dagger) + \frac{1}{2}\partial_i|\vec{\Sigma}|\mathbb{I} - \frac{i}{2}J_i\big) = 0. \tag{A.3.13}$$

To proceed any further we need to right multiply by Ψ, it will be necessary to divide by Σ which is why we require that Ψ is not null. This turns equation (A.3.11) into

$$\mathbf{t}^i\big((\partial_i\Psi)|\Psi|^2 + \Psi(\partial_i\Psi^\dagger)\Psi + \frac{1}{2}(\partial_i|\vec{\Sigma}|)\Psi - \frac{i}{2}J_i\Psi\big) = 0, \tag{A.3.14}$$

which can be further simplified using

$$\begin{aligned}(\partial_i\Psi^\dagger)\Psi &= -\partial_i|\vec{\Sigma}| - \Psi^\dagger\partial_i\Psi = -\partial_i|\vec{\Sigma}| + \frac{1}{2}\partial_i|\vec{\Sigma}| - i\mathrm{Im}(\Psi^\dagger\partial_i\Psi)\\ &= -\frac{1}{2}\partial_i|\vec{\Sigma}| - i\mathrm{Im}(\Psi^\dagger\partial_i\Psi),\end{aligned} \tag{A.3.15}$$

to give

$$\mathbf{t}^i\big(- |\vec{\Sigma}|\partial_i - i\mathrm{Im}(\Psi^\dagger\partial_i\Psi) - \frac{i}{2}J_i\big)\Psi = 0, \tag{A.3.16}$$

$$\Rightarrow \mathbf{t}^i\big(\partial_i + \frac{i}{|\vec{\Sigma}|}(\mathrm{Im}(\Psi^\dagger\partial_i\Psi) + \frac{1}{2}J_i)\big)\Psi = 0. \tag{A.3.17}$$

This is in the form of a Dirac equation

$$\mathbf{t}^i(\partial_i + iA_i)\Psi = 0, \tag{A.3.18}$$

with the gauge potential being given by

$$A_i = \frac{1}{|\vec{\Sigma}|}\big(\frac{1}{2}\varepsilon_i{}^{jk}\partial_j\Sigma_k + \mathrm{Im}(\Psi^\dagger\partial_i\Psi)\big), \tag{A.3.19}$$

□

This result can also be interpreted, treating Σ as a one form, as

$$A_\Psi = -i \star \frac{d\left(\Psi^\dagger d\vec{x} \cdot \vec{\mathfrak{t}}\Psi\right)}{\Psi^\dagger \Psi} - \frac{\mathrm{Im}\left(\Psi^\dagger d\Psi\right)}{\Psi^\dagger \Psi}. \tag{A.3.20}$$

Next we present the Lorentzian version of Lemma 3.9 from Chapter 3. This result is more involved than the previous one and again all the details will be given.

Lemma A.4

Take (Φ, A) a vortex configuration on $SU(1,1)$ and Ψ_H the zero mode of the Dirac operator on $\mathbb{R}^{1,2}$ corresponding to this vortex configuration. Then the spin density of Ψ_H is divergenceless and the gauge potential corresponding to it through the Lorentzian Loss and Yau formula is given by

$$A_{\Psi_H} = -H^* A - \frac{3}{4} H^* \varsigma^0. \tag{A.3.21}$$

Again the proof is very similar to the corresponding Euclidean result but there are several points where we need to be very careful such as when dealing with $\star^2$ on a one-form on $\mathbb{R}^{1,2}$ which due to our sign conventions is the identity. Care is also needed when taking the adjoint of the spinor Ψ_H. Now

$$\Psi_H = \Omega^{-1} G \begin{pmatrix} H^*\Phi \\ 0 \end{pmatrix} \in \mathbb{C}^{1,1} \tag{A.3.22}$$

so taking the adjoint results in

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \mapsto \psi^\dagger = \left(\bar{\psi}^1, -\bar{\psi}^2\right), \tag{A.3.23}$$

and in particular

$$\Psi_H^\dagger = \Omega^{-1} \left(H^*\bar{\Phi}, 0\right) G^{-1}. \tag{A.3.24}$$

Proof. The Lorentzian analogue of the Loss and Yau result, Lemma A.3 above, gives the gauge potential corresponding to Ψ_H as in Equation (A.3.19) to be

$$A_{\Psi_H} = -i \star \frac{d\left(\Psi_H^\dagger d\vec{x} \cdot \vec{\mathfrak{t}}\Psi_H\right)}{\Psi_H^\dagger \Psi_H} - \frac{\operatorname{Im}\left(\Psi_H^\dagger d\Psi_H\right)}{\Psi_H^\dagger \Psi_H} \tag{A.3.25}$$

this is only valid away from the zeros of Φ as that is where Ψ_H vanishes but the singularities will be the same as those in the modulus argument expression for A above. We need to know that the map

$$\star\varsigma^0 \wedge : \Lambda^1(AdS_3) \to \Lambda^1(AdS_3), \varsigma \mapsto i\frac{2}{\ell}\varsigma, \tag{A.3.26}$$

pulls back to a map

$$\vartheta^1 + i\vartheta^2 \mapsto i\frac{2}{\ell}\left(\vartheta^1 + i\vartheta^2\right) \tag{A.3.27}$$

which is written as

$$\frac{2}{\ell} \star_{\mathbb{R}^{1,2}} \vartheta^0 \wedge : \Lambda^1(\mathbb{R}^{1,2}) \to \Lambda^1(\mathbb{R}^{1,2}). \tag{A.3.28}$$

Using the explicit form of Ψ_H and the pullback of the modulus argument decomposition,

$$H^*\Phi = e^{\frac{1}{2}H^*M + iH^*\chi}, \tag{A.3.29}$$

in Equation (A.3.19) we arrive at

$$\begin{aligned} A_{\Psi_H} =& -\frac{1}{2} \star_{\mathbb{R}^{1,2}} \left(\vartheta^0 \wedge H^*M\right) - dH^*\chi \\ & - \begin{pmatrix} 1, 0 \end{pmatrix} \left(i\Omega^2 \star_{\mathbb{R}^{1,2}} d\left(\frac{G^{-1}d\vec{x}\cdot\vec{\mathfrak{t}}G}{\Omega^2}\right) - iG^{-1}dG\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned} \tag{A.3.30}$$

Now we can use that

$$\begin{aligned} \Omega^2 \star_{\mathbb{R}^{1,2}} d\left(\frac{G^{-1}d\vec{x}\cdot\vec{\mathfrak{t}}G}{\Omega^2}\right) &= \Omega^2 \star_{\mathbb{R}^{1,2}} d\left(\frac{H^{-1}dH}{\Omega}\right), \\ &= -H^{-1}dH - \star_{\mathbb{R}^{1,2}}\left(d\Omega \wedge H^{-1}dH\right). \end{aligned} \tag{A.3.31}$$

In the above expression we have used that

$$\star_{\mathbb{R}^{1,2}} d(H^{-1}dH) = -\frac{1}{\Omega}H^{-1}dH, \tag{A.3.32}$$

we can show this as follows (suppressing the subscript on the stars for clarity),

$$\star_{\mathbb{R}^{1,2}} d(H^{-1}dH) = \star H^* \left(d(h^{-1}dh)\right), \tag{A.3.33}$$

$$= \star H^* \left(d\varsigma^i \mathfrak{t}_i\right), \tag{A.3.34}$$

$$= -\frac{1}{2}\varepsilon^i{}_{jk} \star H^* \left(\varsigma^j \wedge \varsigma^k \mathfrak{t}_i\right), \tag{A.3.35}$$

$$= -\frac{1}{2}\varepsilon^i{}_{jk}\frac{\star}{\Omega^2}\vartheta^j \wedge \vartheta^k \mathfrak{t}_i, \tag{A.3.36}$$

$$= -\frac{\star^2}{\Omega^2}\vartheta^i \mathfrak{t}_i, \tag{A.3.37}$$

$$= -\frac{\vartheta^i}{\Omega^2}\mathfrak{t}_i, \tag{A.3.38}$$

$$= -\frac{1}{\Omega}H^{-1}dH, \tag{A.3.39}$$

where we have used that $\star^2\vartheta^i = \vartheta^i$ in our sign conventions, as was mentioned above.

Now from the properties of G and H that were proved in Lemma 4.6 we have that

$$- \star_{\mathbb{R}^{1,2}} \left(d\Omega \wedge H^{-1}dH\right) - G^{-1}dG = -\frac{1}{2}H^{-1}dH, \tag{A.3.40}$$

which means that

$$\Omega^2 \star_{\mathbb{R}^{1,2}} d\left(\frac{G^{-1}d\vec{x}\cdot\vec{\mathfrak{t}}G}{\Omega^2}\right) - G^{-1}dG = -\frac{3}{2}H^{-1}dH. \tag{A.3.41}$$

Finally we can observer that

$$-i\left(1, 0\right)\left(\frac{-3}{2}H^{-1}dH\right)\begin{pmatrix}1\\0\end{pmatrix} = \frac{3i}{2}\left(\frac{i}{2}H^*\varsigma^0\right) = -\frac{3}{4}H^*\varsigma^0 \tag{A.3.42}$$

which results in the claimed expression for A_{Ψ_H}. □

Appendix B

Equivariant functions

Here we present the proof that the diagram in (5.1.6) commutes. It can be encapsulated in the following lemma

Lemma B.1

Consider the equivariant function over $\mathbb{H}^1_\lambda$

$$C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})_N = \{F : \mathbb{H}^1_\lambda \to \mathbb{C} | 2iX_0F = NF\}, \quad N \in \mathbb{Z}. \tag{B.0.1}$$

Using the, possibly local, section

$$s : U \simeq \mathbb{C} \subset M_\lambda \to \mathbb{H}^1_\lambda, \quad z \mapsto \frac{1}{\sqrt{1-\lambda|z|^2}} \begin{pmatrix} 1 & \lambda\bar{z} \\ z & 1 \end{pmatrix}, \tag{B.0.2}$$

of the circle bundle, $\pi : (z_1, z_2) \mapsto \frac{z_2}{z_1}$ *we can establish the relationship*

$$s^* \left(X_+F\right)(z) = i\left(q\bar{\partial} - \lambda\frac{N}{2}z\right) s^*\left(F\right)(z), \tag{B.0.3}$$

where $q = (1-\lambda|z|^2)$.

The proof of this is a direct computation of the same form as that of Lemma A.1 above.

Proof. For an equivariant function, $F \in C^\infty(\mathbb{H}^1_\lambda, \mathbb{C})$ we have that

$$i\left(q\bar{\partial} - \lambda\frac{N}{2}z\right) s^*F = i\left(q\bar{\partial}s^*F - \lambda z s^*\left(iX_0F\right)\right). \tag{B.0.4}$$

Next we have that

$$q\bar{\partial} = \frac{1}{2\sqrt{q}}\left(\lambda z\partial_1 + \lambda z^2\partial_2 + \lambda z\bar{\partial}_1 + 2\bar{\partial}_2 - \lambda|z|^2\partial_2\right), \tag{B.0.5}$$

and

$$izs^*\left(X_0 F\right) = \frac{1}{2\sqrt{q}}\left(z\partial_1 + z^2\partial_2 - z\bar{\partial}_1 - |z|^2\bar{\partial}_2\right)s^*\left(F\right). \tag{B.0.6}$$

Combining these results we have that

$$i\left(q\bar{\partial} - \lambda\frac{N}{2}z\right)s^*\left(F\right)\left(z\right) = \frac{i}{\sqrt{q}}\left(\bar{\partial}_2 + \lambda z\bar{\partial}_1\right)s^*\left(F\right) = s^*\left(X_+ F\right), \tag{B.0.7}$$

which is the desired result. $\square$

Appendix C

Laguerre-Gaussian functions

In this final Appendix we include a discussion of the paraxial Helmholtz equation and its solutions, the Laguerre-Gaussian functions, which we use in Chapter 6 to approximate linked and knotted Rañada fields. The discussion here of the paraxial approximation and Laguerre-Gaussians functions follows that given in [58].

The source free Maxwell equations imply that the electric field $\vec{E}$ solves the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0. \tag{C.0.1}$$

Taking the electric field to have a plane wave form of time dependence $\vec{E} = \vec{\mathcal{E}}e^{-i\omega t}$ this becomes the Helmhotz equation

$$\left(\nabla^2 + k^2\right)\vec{\mathcal{E}} = 0, \tag{C.0.2}$$

with $k = \frac{\omega}{c}$. If we now take the electric field of the laser to be of the form $\mathcal{E}_1 + i\mathcal{E}_2 = \kappa e^{ikz}$ the Helmholtz equation becomes

$$\Delta_{\mathbb{R}^2}\kappa + \frac{\partial^2\kappa}{\partial z^2} + 2ik\frac{\partial\kappa}{\partial z} = 0. \tag{C.0.3}$$

The paraxial approximation is that the variation of κ with respect to z is much smaller than that in x and y and is captured mostly by the e^{ikz}. We thus assume that $\frac{\partial^2\kappa}{\partial z^2}$ is smaller than the other terms and that we can ignore it leading to the paraxial Helmholtz equation

$$\Delta_{\mathbb{R}^2}\kappa + 2ik\frac{\partial \kappa}{\partial z} = 0. \tag{C.0.4}$$

Writing this in the cylindrical coordinates $x = \rho\cos\phi, y = \rho\sin\phi, z$ the solutions are given in terms of Laguerre-Gaussian functions $\mathcal{L}_{pn}(\rho, \phi, z)$ [58]. Adopting the conventions of the supplementary material in [19] the Laguerre-Gaussian functions have the form

$$\begin{aligned}\mathcal{L}_{pn}(\rho, \phi, z) =& \frac{C}{\sqrt{1+\frac{z^2}{z_R^2}}}\left(\frac{\rho\sqrt{2}}{w(z)}\right)^{|n|} L_p^{|n|}\left(\frac{2\rho^2}{w^2(z)}\right) e^{-\frac{\rho^2}{w^2(z)}} \\ &\times e^{-\frac{ik\rho^2 z}{2\left(z^2+z_R^2\right)}} e^{-in\phi} e^{i(2p+|n|+1)\arctan\frac{z}{z_R}},\end{aligned} \tag{C.0.5}$$

with (ρ, ϕ, z) being the cylindrical coordinates, n the azimuthal index giving the angular momentum, p the radial index, C a normalisation constant and $L_p^{|n|}$ are the associated Laguerre polynomials. The beam waist is defined as

$$w(z) = \omega_0\sqrt{1+(z/z_R)^2} \tag{C.0.6}$$

and the Rayleigh range is defined as $z_R = \pi\omega_0^2/\lambda$. For $z = 0$, the Laguerre-Gaussian modes become

$$\mathcal{L}_{pn}(\rho, \phi, 0) = \frac{\tilde{C}}{\omega_0} e^{-\frac{\rho^2}{\omega_0^2}} L_p^n\left(\frac{2\rho^2}{\omega_0^2}\right)\left(\frac{x-iy}{w_0}\right)^n. \tag{C.0.7}$$

In Section 6.3 we use these Laguerre-Gaussian functions to approximate the complex polynomials P_1, P_2 and give Rabi coefficients which encode the linked and knotted structures.

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